Example problems: Stoke's Theorem

1. Evaluate $\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, dS$ where $\vec{F} = -y\vec{i} + (x - 2x^3 z)\vec{j} + xy^3\vec{k}$

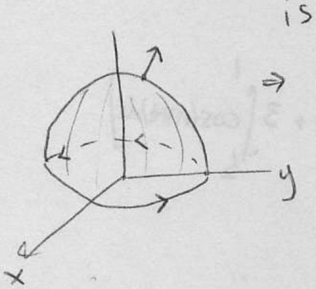
and S = upper hemisphere of radius a centered at $(0,0,0)$

(so ∂S = circle of radius a in xy plane!)

using Stoke's theorem:

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, dS = \oint_{\partial S} \vec{F}\cdot d\vec{R}$$

if we orient S so that $\vec{n}$ is outward, then ∂S (boundary of S) is oriented counter clockwise.



$\Rightarrow$ we need to get a counterclockwise orientation of $x^2 + y^2 = a^2$ $(z=0)$.

$\vec{R}(\theta) = \langle a\cos\theta, a\sin\theta, 0\rangle$ works! $0 \le \theta \le 2\pi$

$$\Rightarrow \frac{d\vec{R}}{d\theta} = \langle -a\sin\theta, a\cos\theta, 0\rangle$$

$$\int_0^{2\pi} \vec{F}\cdot d\vec{R} = \int_0^{2\pi} \vec{F}(\vec{R}(\theta))\cdot\frac{d\vec{R}}{d\theta}\, d\theta = \int_0^{2\pi} \langle -a\sin\theta, a\cos\theta - \cancelto{0}{2a^3\cos^3\theta\cdot 0}, a^4\cos\theta\sin^3\theta\rangle \cdot \langle -a\sin\theta, a\cos\theta, 0\rangle\, d\theta$$

$$= \int_0^{2\pi} a^2\sin^2\theta + a^2\cos^2\theta\, d\theta = a^2\cdot 2\pi$$

2. Suppose we have the same $\vec{F}$ as in example 1, and we still want

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, dS$$

but now S = entire sphere of radius a centered at $(0,0,0)$.

then ∂S is empty! (a sphere has no boundary)

So, we expect that the answer is zero.

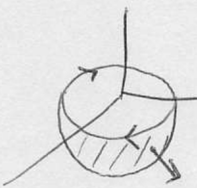
To show this: break the sphere into it's upper + lower hemispheres – denoted S^+ and S^-:

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, dS = \iint_{S^+} (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, dS + \iint_{S^-} (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, dS$$

$$\overset{\text{by Stokes'}}{=} \int_{\partial S^+} \vec{F}\cdot d\vec{R} + \int_{\partial S^-} \vec{F}\cdot d\vec{R} \overset{\text{by \#1}}{=} 2\pi a^2 + \int_{\partial S^-} \vec{F}\cdot d\vec{R} \quad (\star)$$

to find $\int_{\partial S^-} \vec{F}\cdot d\vec{R}$ we need an appropriately oriented parametrization of ∂S^-.

Since the lower hemisphere S^- must also be oriented so that $\vec{n}$ is outward to be consistant with S^+, this induces a <u>clockwise orientation</u> on ∂S^-! So we can use



$$\vec{R}(\theta) = \langle -a\cos\theta, -a\sin\theta, 0\rangle$$

$$\Rightarrow \int_0^{2\pi} \vec{F}\cdot d\vec{R} = \int_0^{2\pi} -a^2\sin^2\theta - a^2\cos^2\theta \, d\theta = -a^2\cdot 2\pi$$

Subbing into $(\star)$

$\Rightarrow$ over the whole sphere S: $\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, dS = 2\pi a^2 - 2\pi a^2 = 0$ ✓

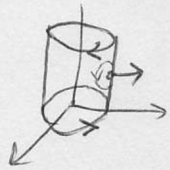
3: Evaluate $\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS$ over the <u>lateral</u> surface

of a cylinder with radius 1 between $z=0$ and $z=1$

where $\vec{F}(x,y,z) = \langle xz, y, y^2\rangle$ and $\vec{n}$ is the outward normal.

By Stoke's thm: $\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS = \int_{\partial S} \vec{F}\cdot d\vec{R}$

What is ∂S here? $\vec{n}$ = outward normal



Now ∂S consists of two pieces - the top rim at $z=1$ (denoted with ∂S_1) and the bottom rim at $z=0$ (∂S_0), and $\vec{n}$ induces the orientations shown on the picture.

$$\int_{\partial S}\vec{F}\cdot d\vec{R} = \int_{\partial S_0}\vec{F}\cdot d\vec{R}_0 + \int_{\partial S_1}\vec{F}\cdot d\vec{R}_1$$

$\vec{R}_0 = \langle\cos\theta, \sin\theta, 0\rangle$ parametrizes ∂S_0 ctr clockwise

$\vec{R}_1 = \langle -\cos\theta, -\sin\theta, 1\rangle$ param of ∂S_1 clockwise

$$= \int_{0\,(\partial S_0)}^{2\pi} \underbrace{\langle\cos\theta\cdot 0, \sin\theta, \sin^2\theta\rangle}_{\vec{F}(\vec{R}_0(\theta))}\cdot\underbrace{\langle-\sin\theta, \cos\theta, 0\rangle}_{d\vec{R}_0/d\theta}\,d\theta$$

$$+ \int_{0\,(\partial S_1)}^{2\pi}\langle-\cos\theta, -\sin\theta, \sin^2\theta\rangle\cdot\langle\sin\theta, -\cos\theta, 0\rangle\,d\theta$$

$$= \int_0^{2\pi}\cos\theta\sin\theta\,d\theta + \int_0^{2\pi} 0\,d\theta = \int_0^{2\pi} u\,du = \frac{1}{2}(\sin^2\theta)\Big|_0^{2\pi} = 0$$

Let $u = \sin\theta$
$du = \cos\theta\,d\theta$

It just so happened that we got zero in this case.

But if I had said integrate over the closed surface bounded by $z=0$, $z=1$ and $x^2+y^2=1$, this surface has no boundary and so:

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS = \int_{\underbrace{\partial S=\{\phi\}}_{\text{empty set!}}} \vec{F}\cdot d\vec{R} = 0$$

Also, seperately, the surface S would bound a region in 3 space - call it V - namely the solid cylindar of radius 1, height 1, so we can apply the divergence theorem!

V = solid cylindar, $\partial V = S$

$$\iint_{S=\partial V} (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS = \iiint_V \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F})\,dV$$

but several homeworks ago, you showed that $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F})=0$ always! (as long as $\vec{F}$ is smooth enough)

$\Rightarrow$ Again we see that $\displaystyle\iint_{S=\partial V} (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS = 0$

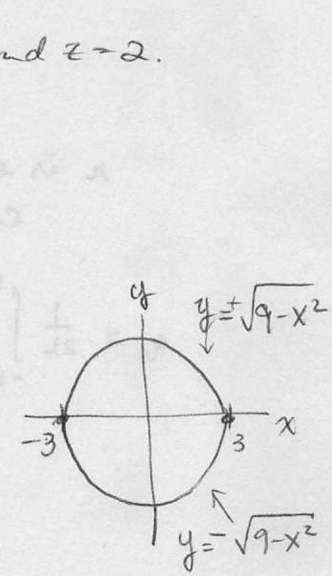
but now because S is a boundary of a 3D volume.

Some divergence theorem examples:

Let V = solid cylindar of radius 3 between $z=0$ and $z=2$.

Find $\displaystyle\iint_{\partial V} \vec{F}\cdot\vec{n}\, dS$ using the divergence theorem.

First way: By divergence thm: $\displaystyle\iint_{\partial V} \vec{F}\cdot\vec{n}\, dS = \iiint_V \vec{\nabla}\cdot\vec{F}\, dV$



we can describe the solid cylindar as:

$$0 \le z \le 2, \quad -3 \le x \le 3, \quad -\sqrt{9-x^2} \le y \le +\sqrt{9-x^2}$$

So $$\iiint_V \vec{\nabla}\cdot\vec{F}\, dV = \int_0^2 \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{+\sqrt{9-x^2}} \vec{\nabla}\cdot\vec{F}\, dy\, dx\, dz$$

Say $\vec{F} = x\vec{i} + (x+y)\vec{j} + (x+y+z)\vec{k} \Rightarrow \vec{\nabla}\cdot\vec{F} = 3$

$$= \int_0^2 \int_{-3}^{3} 3\left(\sqrt{9-x^2} + \sqrt{9-x^2}\right) dx\, dz = 6\int_0^2 \int_{-3}^{3} \sqrt{9-x^2}\, dx\, dz$$

unless you are comfortable with trig substitution, this is not an easy integral.

So what can we do?

Second Way: Use cylindrical coordinates! $\langle x, y, z\rangle = \langle r\cos\theta, r\sin\theta, z\rangle = \vec{R}(r,\theta,z)$

with $0 \le \theta < 2\pi$, $0 \le r \le 3$, $0 \le z \le 2$

The question is - how does dV look in these coordinates??

determinant of matrix

$$dV = \left| \begin{matrix} — \partial\vec{R}/\partial r — \\ — \partial\vec{R}/\partial\theta — \\ — \partial\vec{R}/\partial z — \end{matrix} \right| = \left| \begin{matrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{matrix} \right| = \cos\theta(r\cos\theta) + \sin\theta(r\sin\theta) + 0$$

$$= r$$

so $\underline{dV = r\,dr\,d\theta\,dz}$ in cylindrical coords.

$$\iiint_V \vec{\nabla}\cdot\vec{F}\,dV = \iiint_V 3\,dV = \int_0^2\int_0^{2\pi}\int_0^3 3r\,dr\,d\theta = \int_0^2\int_0^{2\pi} \frac{3}{2}r^2\Big|_0^3 \,d\theta\,dz$$

$$= \frac{27}{2}\int_0^2\int_0^{2\pi} d\theta\,dz = 4\pi\cdot\frac{27}{2} = 54\pi$$

Example 2: Let $\vec{F} = \phi \vec{\nabla} \phi$, $\phi = x + y + z$

S = sphere at $\vec{0}$ of radius 3

Use div. theorem to compute $\iint_S \vec{F} \cdot \vec{n} \, dS$:

$$\iint_S \vec{F} \cdot \vec{n} \, ds = \iiint_V \vec{\nabla} \cdot \vec{F} \, dV = \iiint_V \vec{\nabla} \cdot (\phi \vec{\nabla} \phi) = \iiint_V (\vec{\nabla}\phi \cdot \vec{\nabla}\phi + \phi \Delta \phi) \, dV$$

since $\vec{\nabla}\phi = \langle 1, 1, 1 \rangle$ $(\Rightarrow \Delta\phi = 0)$

$$= \iiint_V 3 \, dV = 3 \iiint_V dV = 3 \text{ volume of } B(\vec{0}, 3)$$

$$= 3 \cdot \frac{4}{3} \pi (3)^3 = (27)(4\pi)$$