Exam 3 Solutions:

1 (a) Let $\vec{F}$ be a continuously differentiable vector field on a bounded domain (3D) V, and let S be the closed surface bounding V, then

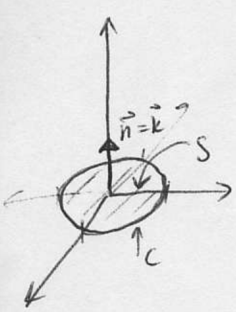
$$\iiint_V \vec{\nabla}\cdot\vec{F}\, dV = \iint_S \vec{F}\cdot d\vec{S}$$

(b) If $\vec{F}$ continuously differentiable vector field on a bounded surface S and C is the boundary of S, then

$$\iint_S \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{R}$$

2. (a) normal flux of $\vec{F}$ through S $= \iint_S \vec{F}\cdot\vec{n}\, dS$

parametrizing S, we get:



$$\vec{R}(r,\theta) = \langle r\cos\theta, r\sin\theta, 0\rangle \qquad 0 \le r \le 1 \quad 0 \le \theta < 2\pi$$

$$d\vec{S} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} dr\,d\theta = -r\,k = \langle 0,0,-r\rangle\, dr\,d\theta$$

$$\Rightarrow dS = |d\vec{S}| = r\,dr\,d\theta$$

$$\iint_S \vec{F}\cdot\vec{n}\, dS = \int_0^{2\pi}\int_0^1 \langle r^2\cos^2\theta, 2r^2\cos\theta\sin\theta, 0\rangle\cdot\langle 0,0,1\rangle\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 0\, dr\,d\theta = 0$$

(b) $$\oint_C \vec{F}\cdot d\vec{R} = \int_0^{2\pi} \langle \cos^2\theta, 2\cos\theta\sin\theta, 0\rangle\cdot\langle -\sin\theta, \cos\theta, 0\rangle\, d\theta$$

$\vec{R}(\theta) = \langle \cos\theta, \sin\theta, 0\rangle$

$$= \int_0^{2\pi} \cos^2\theta\sin\theta\, d\theta \qquad \text{Let } u=\cos\theta,\ du = -\sin\theta\, d\theta$$

$$= \int_0^{2\pi} -u^2\, du = -\frac{1}{3}\cos^3\theta\Big|_0^{2\pi} = -\frac{1}{3}[1-1] = 0$$

(c) $\iint_S \vec{\nabla}\times\vec{F}\cdot\vec{n}\,dS = ?$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & 2xy & z \end{vmatrix} = 0\vec{i} + 0\vec{j} + 2y\vec{k} = \langle 0, 0, 2y\rangle$$

from (a) $dS = r\,dr\,d\theta$

$\vec{R}(r,\theta) = \langle r\cos\theta, r\sin\theta, 0\rangle$

$\vec{n} = \vec{k}$

$$\int_0^{2\pi}\int_0^1 \langle 0,0,2r\sin\theta\rangle\cdot\langle 0,0,1\rangle\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2r^2\sin\theta\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{2}{3}r^3\Big|_0^1 \sin\theta\,d\theta = \frac{2}{3}\int_0^{2\pi}\sin\theta\,d\theta = -\frac{2}{3}\cos\theta\Big|_0^{2\pi} = -\frac{2}{3}[1-1] = 0$$

#3) (a) By the divergence theorem

$$\iiint_V \vec{\nabla}\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot d\vec{S}$$

$\vec{F} = \langle 3x, y, z\rangle \Rightarrow \vec{\nabla}\cdot\vec{F} = 3+1+1 = 5$

So

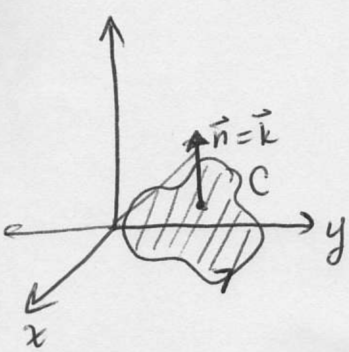
$$\iiint_V \vec{\nabla}\cdot\vec{F}\,dV = \iiint_V 5\,dV = 5\cdot\text{Volume of } V = 5\cdot 10 = 50$$

$$\parallel$$

$\iint_S \vec{F}\cdot d\vec{S}$ = net flux of $\vec{F}$ through $S = \partial V$.

(b) If C is a simple closed curve in the xy plane $(z=0)$ then the region it bounds in the xy plane has normal $\vec{n} = \vec{k}$, since C is oriented counterclockwise.

Stoke's theorem says: $\iint_S \vec{\nabla}\times\vec{F}\cdot\vec{n}\,dS = \oint_{C=\partial S} \vec{F}\cdot d\vec{R}$



If $\vec{\nabla}\times\vec{F} = \vec{k}$, since $\vec{n} = \vec{k}$,

$$\iint_S \vec{\nabla}\times\vec{F}\cdot\vec{n}\,dS = \iint_S \vec{k}\cdot\vec{k}\,dS = \iint_S dS = \text{area of region } S.$$

and so $\boxed{\text{area of } S = \oint_{C=\partial S} \vec{F}\cdot d\vec{R} \quad \text{if } \vec{\nabla}\times\vec{F} = \vec{k}}$

(4) The boundary of the surface S (which is like a silo without a bottom) is the circle $x^2+y^2=1$ in the xy plane $(z=0)$. By Stoke's theorem

$$\iint_S \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \int_{\partial S} \vec{F}\cdot d\vec{R}$$

So parameterizing the boundary of S (∂S) as

$$\vec{R}(\theta) = \langle \cos\theta, \sin\theta, 0\rangle \qquad 0 \le \theta < 2\pi$$

we see

$$\int_{\partial S} \vec{F}\cdot d\vec{R} = \int_0^{2\pi} \underbrace{\langle \cos\theta, \sin\theta, 0\rangle}_{\vec{F}(\vec{R}(\theta))} \cdot \frac{d\vec{R}}{d\theta}\, d\theta$$

$$= \int_0^{2\pi} \langle \cos\theta, \sin\theta, 0\rangle \cdot \langle -\sin\theta, \cos\theta, 0\rangle\, d\theta$$

$$= \int_0^{2\pi} (-\sin\theta\cos\theta + \sin\theta\cos\theta)\, d\theta = 0$$

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(5) We want to show that $\iiint_V q\, dV = 0$:

since $q = \Delta f$:

$$\iiint_V q\, dV = \iiint_V \Delta f\, dV = \iiint_V \vec{\nabla}\cdot\vec{\nabla} f\, dV \overset{\text{because } \vec{F}=\vec{\nabla}f}{=} \iiint_V \vec{\nabla}\cdot\vec{F}\, dV$$

by the divergence theorem

$$= \iint_{S=\partial V} \vec{F}\cdot\vec{n}\, dS \underset{b/c\ \vec{F}=\vec{\nabla}f}{=} \iint_S \vec{\nabla}f\cdot\vec{n}\, dS = \iint_S \frac{\partial f}{\partial \vec{n}}\, dS$$

$\Rightarrow$ since $\frac{\partial f}{\partial \vec{n}}$ is zero on the boundary of V $(\partial V = S)$ then $\iint_S \frac{\partial f}{\partial \vec{n}}\, dS = 0 \Rightarrow \iiint_V q\, dV = 0$ //