Exam 1 Solutions:

#1 (a) l_1: $x = 2+t$, $y = 4+t$, $z = 6+2t$

$\Rightarrow t = x-2,\ t = y-4,\ t = \frac{z-6}{2}$

$\Rightarrow \boxed{x-2 = y-4 = \frac{z-6}{2}}$ nonparam. form

l_2: $x = -1+2t$, $y = 3$, $z = 7-3t$

$\Rightarrow t = \frac{x+1}{2},\ y = 3,\ t = \frac{z-7}{-3}$

$\Rightarrow \boxed{\frac{x+1}{2} = \frac{z-7}{-3},\ y = 3}$ non-param. form

at the intersection of l_1 & l_2, the "x"'s, "y's", and "z"'s are the same. Since $y=3$ always in line 2, then at the intersection point it must be that $y=3$.

then in l_1 we have:

$$x-2 = -1 = \frac{z-6}{2} \Rightarrow x = 1,\ z = 4$$

So they intersect at $(1,3,4)$ if this also satisfies l_2:

Check $x=1, z=4$:

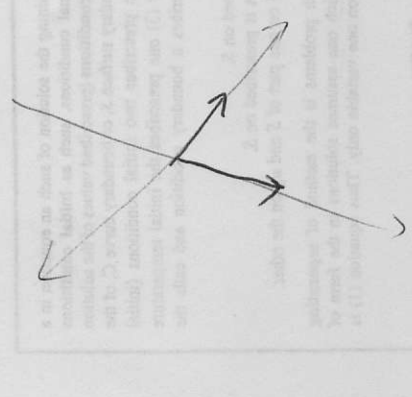
$$\frac{1+1}{2} \stackrel{?}{=} \frac{4-7}{-3} \Rightarrow 1 = 1 \checkmark$$

(b) direction for l_1: $\langle 1,1,2\rangle$

" " l_2: $\langle 2,0,-3\rangle$

$\langle 1,1,2\rangle \cdot \langle 2,0,-3\rangle = -4 \neq 0$

since the dot product is not zero $\Rightarrow$ not $\perp$.



(c) If ℓ_1 and ℓ_2 form a plane, then their direction vectors lie in the plane. Also their point of intersection $(1,3,4)$ lies in the plane. To get a normal to the plane we take:

$$\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ 2 & 0 & -3 \end{vmatrix} = -3\vec{i} + 7\vec{j} - 2\vec{k}$$

and so the plane is:

$$\langle -3, 7, -2 \rangle \cdot \langle x, y, z \rangle = \langle -3, 7, -2 \rangle \cdot \langle 1, 3, 4 \rangle$$

$$-3x + 7y - 2z = -3 + 21 - 8 = 10$$

#2 (a)

$$\vec{T}(t) = \frac{d\vec{R}/dt}{|d\vec{R}/dt|} = \frac{\langle -\pi \sin \pi t, \pi \cos \pi t \rangle}{|\langle -\pi \sin \pi t, \pi \cos \pi t \rangle|}$$

$$= \frac{\langle -\pi \sin \pi t, \pi \cos \pi t \rangle}{\sqrt{\pi^2 \underbrace{(\sin^2 \pi t + \cos^2 \pi t)}_{=1}}} = \langle -\sin \pi t, \cos \pi t \rangle$$

(b)

$$\vec{N}(t) = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = \frac{\langle -\pi \cos \pi t, -\pi \sin \pi t \rangle}{\sqrt{\pi^2 (\cos^2 \pi t + \sin^2 \pi t)}} = \langle -\cos \pi t, -\sin \pi t \rangle$$

(c)

$$a_t = \frac{d}{dt}(|\vec{v}|) = \frac{d}{dt}(\pi) = 0 \quad , \quad a_n = \left(\frac{ds}{dt}\right)^2 \left|\frac{d\vec{T}}{ds}\right| = \left(\frac{ds}{dt}\right)^2 \frac{|d\vec{T}/dt|}{|ds/dt|}$$

$$\left[\text{ in (a) } |\vec{v}| = \left|\frac{d\vec{R}}{dt}\right| = \pi \right]$$

$$= |ds/dt| \, |d\vec{T}/dt|$$

$$= \pi \, |\langle -\pi \cos \pi t, -\pi \sin \pi t \rangle|$$

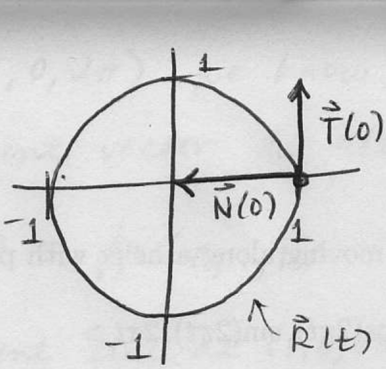
$$= \pi \cdot \pi = \pi^2$$

<u>OR</u>: $\vec{a} = \langle -\pi^2 \cos \pi t, -\pi^2 \sin \pi t \rangle = \pi^2 \vec{N}$

so: $|\text{proj}_{\vec{T}}(\vec{a})| = \left| \frac{\vec{a}\cdot\vec{T}}{\vec{T}\cdot\vec{T}} \vec{T} \right| = \left| \frac{\pi^2 \vec{N}\cdot\vec{T}}{\vec{T}\cdot\vec{T}} \vec{T} \right| = 0$ since $\vec{N} \perp \vec{T}$.

$|\text{proj}_{\vec{N}}(\vec{a})| = \left| \frac{\vec{a}\cdot\vec{N}}{\vec{N}\cdot\vec{N}} \vec{N} \right| = \left| \frac{\pi^2 \vec{N}\cdot\vec{N}}{\vec{N}\cdot\vec{N}} \right| |\vec{N}| = \pi^2$ since $|\vec{N}| = 1$ and $\vec{N}\cdot\vec{N} = |\vec{N}|^2$

(d)



$\vec{T}(0) = \langle 0, 1 \rangle$

$\vec{N}(0) = \langle -1, 0 \rangle$

(e) yes – The motion is circular, and the speed $|\vec{v}| = \pi$ is constant, so the accleration is entirely centripetal, or entirely in the direction of the normal $\vec{N}$.

#3 (a)

$$\vec{T}(t) = \frac{\langle -2\pi \sin(2\pi t),\ 2\pi\cos(2\pi t),\ 2\pi \rangle}{\sqrt{(-2\pi\sin(2\pi t))^2 + (2\pi\cos(2\pi t))^2 + (2\pi)^2}} = \frac{d\vec{R}/dt}{|d\vec{R}/dt|}$$

$$= \frac{\langle -2\pi\sin(2\pi t),\ 2\pi\cos(2\pi t),\ 2\pi \rangle}{\sqrt{4\pi^2\underbrace{(\sin^2(2\pi t) + \cos^2(2\pi t))}_{=1} + 4\pi^2}}$$

$$= \frac{\langle -2\pi\sin(2\pi t),\ 2\pi\cos(2\pi t),\ 2\pi \rangle}{\sqrt{4\pi^2 + 4\pi^2}} \qquad = 2\sqrt{2}\,\pi$$

$$= \left\langle -\frac{1}{\sqrt{2}}\sin(2\pi t),\ \frac{1}{\sqrt{2}}\cos(2\pi t),\ \frac{1}{\sqrt{2}} \right\rangle$$

(b)

$$k = \left|\frac{d\vec{T}}{ds}\right| = \frac{|d\vec{T}/dt|}{|ds/dt|} = \frac{|\langle -\sqrt{2}\pi\cos(2\pi t), -\sqrt{2}\pi\sin(2\pi t), 0\rangle|}{2\sqrt{2}\,\pi}$$ ← from above

$$= \frac{\sqrt{2\pi^2}}{2\sqrt{2}\,\pi} = \frac{1}{2}$$

(c)

$$\text{arclength} = \int_{t_1}^{t_2} \left|\frac{d\vec{R}}{dt}\right| dt = \int_0^1 2\sqrt{2}\,\pi\, dt = 2\sqrt{2}\,\pi$$

$t_1 = 0$ since $\vec{R}(0) = (1,0,0)$

$t_2 = 1$ since $\vec{R}(1) = (1,0,2\pi)$

#3 (d) at $(1,0,2\pi)$ we know $t=1$.

the tangent vector to $\vec{R}(t)$ at $t=1$ is

$$\vec{T}(1) = \frac{1}{\sqrt{2}}\langle 0,1,1\rangle$$

the tangent line at $(1,0,2\pi)$ has as it's direction $\frac{1}{\sqrt{2}}\langle 0,1,1\rangle$

so the line is given by:

$$\langle 1,0,2\pi\rangle + \frac{1}{\sqrt{2}}\langle 0,1,1\rangle t = \langle 1, \frac{1}{\sqrt{2}}t, 2\pi + \frac{1}{\sqrt{2}}t\rangle$$

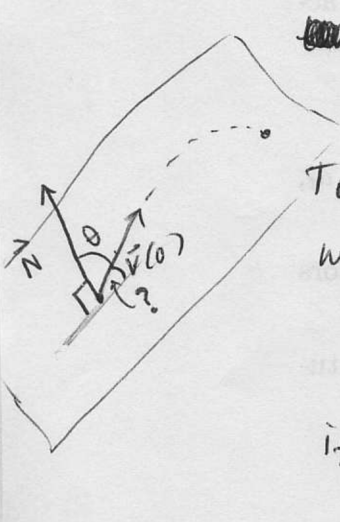
or
$$x(t) = 1$$
$$y(t) = \frac{1}{\sqrt{2}}t$$
$$z(t) = 2\pi + \frac{1}{\sqrt{2}}t$$

#4 (a)
$$\vec{v}(t) = \frac{d\vec{R}}{dt} = \langle 1, 1, 3-2t\rangle$$
$$\vec{a}(t) = \frac{d^2\vec{R}}{dt^2} = \langle 0,0,-2\rangle$$

(b) $\vec{v}(0) = \langle 1,1,3\rangle$

a normal to the plane is $\langle -1,-1,2\rangle$

since the equation of the plane is $-x-y+2z=0$.



To find the angle between $\vec{v}(0)$ and the plane, we can do:

$$(\measuredangle\ \vec{v}(0) \text{ to plane}) = 90° - (\measuredangle\ \vec{v}(0) \text{ to normal})$$

if $\theta = \measuredangle\ \vec{v}(0)$ to normal

$$\Rightarrow \cos\theta = \frac{\vec{v}(0)\cdot\vec{N}}{|\vec{v}(0)||\vec{N}|} = \frac{\langle 1,1,3\rangle\cdot\langle -1,-1,2\rangle}{\sqrt{11}\cdot\sqrt{6}}$$

$$= \frac{4}{\sqrt{66}} \Rightarrow \theta = \arccos\left(\frac{4}{\sqrt{66}}\right)$$

and our answer is $90° - \arccos\left(\frac{4}{\sqrt{66}}\right)$

(c) at $t=1$, the ball's position is $\vec{R}(1) = \langle 1,1,2 \rangle$

thus the distance from $\langle 1,1,2 \rangle$ to the plane $-x-y+2z=0$ is:

$$\frac{(\langle 1,1,2\rangle - \langle 0,0,0\rangle)\cdot \vec{N}}{|\vec{N}|} = \frac{\langle 1,1,2\rangle \cdot \vec{N}}{|\vec{N}|} = \frac{\langle 1,1,2\rangle \cdot \langle -1,-1,2\rangle}{\sqrt{6}}$$

$$= \frac{2}{\sqrt{6}}$$

(d) notice the unit tangent vector

$$\vec{T}(t) = \frac{\langle 1, 1, 3-2t \rangle}{\sqrt{2+(3-2t)^2}}$$

is not constant! So the path cannot be a straight line.