Solution for HW #3.

· Set 10.4

· <u>Problem 3</u>: Function is neither even or odd.

. pick point $x_0 = \frac{\pi}{2}$. , since $f(x)$ is of period 2π and $f(x) = x^2$ if $0 < x < 2\pi$,

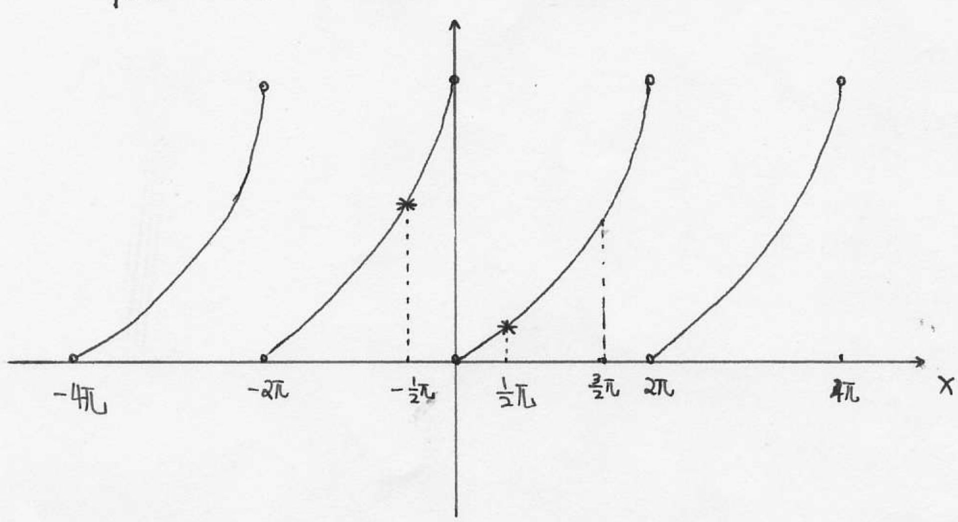
$$f(-x_0) = f(-\tfrac{\pi}{2}) = f(-\tfrac{\pi}{2} + 2\pi) = f(\tfrac{3}{2}\pi) = (\tfrac{3}{2}\pi)^2 \quad [0 < \tfrac{3}{2}\pi < 2\pi].$$

Thus $f(x_0) = (\frac{\pi}{2})^2 \neq (\frac{3}{2}\pi)^2 = f(-x_0)$.

$f(x)$ is not even.

And $f(x_0) = (\frac{\pi}{2})^2 \neq -(\frac{3}{2}\pi)^2 = -f(-x_0)$

$f(x)$ is not odd.



· **Problem 5:** Function is even.

(1) If $-\pi < x < \pi$, i.e. $x \in (-\pi, \pi)$

then $-\pi < -x < \pi$, i.e. $-x \in (-\pi, \pi)$

Thus $f(-x) = e^{-|-x|} = e^{-|x|} = f(x)$

(2) If $x \notin (-\pi, \pi)$

and (*) $x \neq (2k+1)\pi$ $k \in \mathbb{Z}$ (i.e. $k = \cdots, -3, -2, -1, 0, 1, 2, \cdots$)

((*). We don't consider these points, because there is no definition on these points.)

There exists an integer n. (i.e. $n \in \mathbb{Z}$). such that

$$x + n \cdot (2\pi) \in (-\pi, \pi)$$

and $$-x - n \cdot (2\pi) \in (-\pi, \pi)$$

Thus $$f(x) = f(x + n \cdot (2\pi)) = e^{-|x+n \cdot (2\pi)|}$$

$$= e^{-|-(x+n\cdot(2\pi))|}$$

$$= f(-x - n \cdot 2\pi)$$

$$= f(-x)$$

From (1) and (2) we know that $f(x)$ is <u>even</u>.

Problem 11 Function is even, with period 2π.

Note Function is also odd if $k = 0$.

Write the function on interval $[-\pi, \pi]$:

$$f(x) = \begin{cases} 0 & , \text{ if } -\pi \le x < -\pi/2 \\ k & , \text{ if } -\pi/2 < x < \pi/2 \\ 0 & , \text{ if } \pi/2 < x \le \pi \end{cases}$$

Following the discussion like problem 5, we can prove the function is even.

Since function is even, Fourier series of $f(x)$ is Fourier cosine series, namely $b_n = 0$.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi/2} k\, dx$$

$$= k/2$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi/2} k \cos nx \, dx$$

$$= \frac{2k}{n\pi} \sin \frac{n\pi}{2}$$

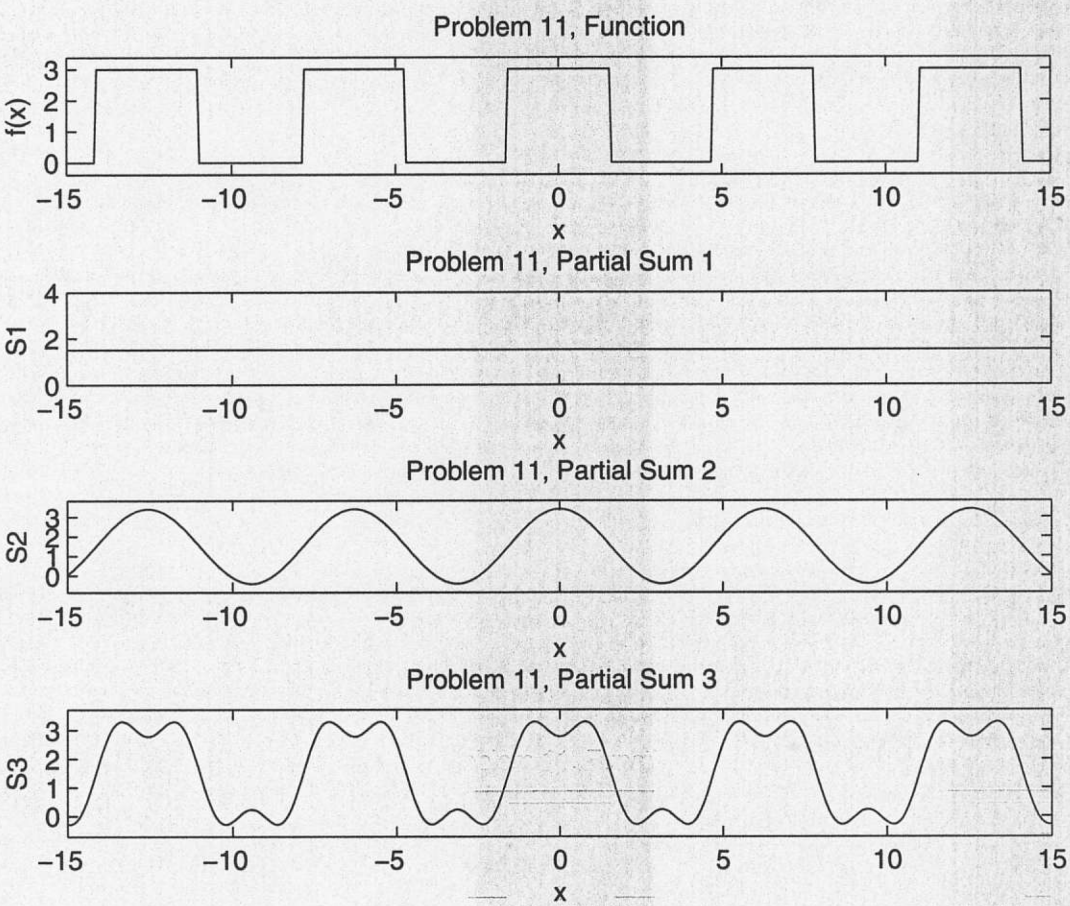
OR: $$= \begin{cases} \frac{2k}{n\pi} & \text{if} \quad n = 4m - 3 \quad m \in \mathbb{N} \\ 0 & \text{if} \quad n = 2m \quad m \in \mathbb{N} \\ -\frac{2k}{n\pi} & \text{if} \quad n = 4m - 1 \quad m \in \mathbb{N} \end{cases}$$

OR: $$= \begin{cases} (-1)^{\frac{n-1}{2}} \cdot \frac{2k}{n\pi} \, , & \text{if} \quad n = 2m-1 \quad m \in \mathbb{N} \\ 0 \, , & \text{if} \quad n = 2m \quad m \in \mathbb{N}. \end{cases}$$

So

$$f(x) = k/2 + \sum_{n=1}^{\infty} \frac{2k}{n\pi} \sin \frac{n\pi}{2} \cos nx$$

OR: $$= k/2 + \sum_{m=1}^{\infty} \frac{2k}{(2m-1)\pi} \cdot (-1)^{m-1} \cdot \cos((2m-1)x)$$


Problem 11, Function
f(x)
x
Problem 11, Partial Sum 1
S1
x
Problem 11, Partial Sum 2
S2
x
Problem 11, Partial Sum 3
S3
x

Problem 18 : From Problem 15 ,

We know that the Fourier coefficients of function in problem 15 are :

$$a_0 = \frac{\pi^2}{6}$$

$$a_n = \frac{2}{n^2} \cos n\pi$$

$$b_n = 0$$

So the Fourier series is

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2} \cos n\pi \cdot \cos nx$$

$$= \frac{\pi^2}{6} + 2\left(-1 \cdot \cos x + \frac{1}{4} \cos 2x - \frac{1}{9} \cos 3x + \frac{1}{16} \cos 4x - \frac{1}{25} \cos 5x + \cdots\right)$$

Choose $x = \pi$ (Note: There is no definition on π, but we can extend it.)

$$f(\pi) = \frac{\pi^2}{2} = \frac{\pi^2}{6} + 2\left(-1 \cdot (-1) + \frac{1}{4} \cdot 1 - \frac{1}{9} \cdot (-1) + \frac{1}{16} \cdot 1 - \frac{1}{25} \cdot (-1) + \cdots\right)$$

$$= \frac{\pi^2}{6} + 2\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots\right)$$

$$\Rightarrow \quad \frac{\pi^2}{2} - \frac{\pi^2}{6} = 2\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots\right)$$

$$\Rightarrow \quad \frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$$

Problem 23

- Fourier cosine series.

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} (\pi - x)\,dx$$

$$= \frac{1}{\pi} \cdot \pi x - \frac{x^2}{2}\Big|_0^{\pi}$$

$$= \frac{1}{\pi} \cdot \left(\pi^2 - \frac{1}{2}\pi^2\right)$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi} \pi \cos nx\,dx - \int_0^{\pi} x \cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{n}\sin nx\Big|_0^{\pi} - \frac{1}{n}\left(x \cdot \sin nx\Big|_0^{\pi} - \int_0^{\pi} \sin nx\,dx\right)\right]$$

$$= \frac{2}{\pi}\left[0 - 0 + \frac{1}{n}\int_0^{\pi} \sin nx\,dx\right]$$

$$= -\frac{2}{n^2\pi} \cos nx\Big|_0^{\pi}$$

$$= \frac{2}{n^2\pi}(1 - \cos n\pi)$$

$$= \begin{cases} \dfrac{4}{n^2\pi} & \text{if} \quad n = 2m-1 \quad m \in \mathbb{N} \\ 0 & \text{if} \quad n = 2m \quad m \in \mathbb{N}. \end{cases}$$

So Fourier cosine series is :

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi} (1-\cos n\pi) \cdot \cos nx$$

$$= \frac{\pi}{2} + \sum_{m=1}^{\infty} \frac{4}{(2m-1)^2 \cdot \pi} \cos((2m-1)x)$$

. Fouries sine series :

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi - x) \sin nx \, dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi} \pi \sin nx \, dx - \int_0^{\pi} x \sin nx \, dx \right]$$

$$= \frac{2}{\pi} \left[-\frac{\pi}{n} \cos nx \Big|_0^{\pi} + \frac{1}{n} \left(x \cos nx \Big|_0^{\pi} - \int_0^{\pi} \cos nx \, dx \right) \right]$$

$$= \frac{2}{n} [1 - \cos n\pi] + \frac{2}{n} \cos n\pi - \frac{2}{n^2\pi} \sin nx \Big|_0^{\pi}$$

$$= \frac{2}{n}$$

So Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n} \sin nx$$

· Set 10.6

Problem 4

Consider the differential equations:

$$y'' + \omega^2 y = \cos \alpha t \quad \text{---- (1)}$$

&

$$y'' + w^2 y = \cos \beta t \quad \text{---- (2)}$$

Solution of equation (1) has the form

$$y_1 = A_1 \cos \alpha t + B_1 \sin \alpha t$$

By substituting this into (1), we find that

$$A_1 = \frac{w^2 - 1}{\alpha^2} \qquad B_1 = \frac{w^2}{\alpha^2}$$

Similarly, solution of equation (2) is

$$y_2 = A_2 \cos \beta t + B_2 \sin \beta t$$

where $A_2 = \frac{w^2 - 1}{\beta^2}$, $B_2 = \frac{w^2}{\beta^2}$.

So by the linearity of the equation,

the solution is

$$y = A_1 \cos \alpha t + B_1 \sin \alpha t + A_2 \cos \beta t + B_2 \sin \beta t$$

Problem 7

Represent $r(t)$ by a Fourier series :

$$r(t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}(1-\cos n\pi)\cos nt$$

$$= \frac{\pi}{2} + \sum_{m=1}^{\infty} \frac{4}{(2m-1)^2\pi} \cos((2m-1)t)$$

(Using the result for Fourier cosine series in Prolem 23, set 10.4)

Consider the equation

$$y_0'' + w^2 y_0 = \frac{\pi}{2}$$

$Y_0 = \frac{\pi}{2 \cdot w^2}$ is the solution

Consider the equation

(*) $$y'' + w^2 y = \frac{4}{n^2\pi} \cos nt \qquad (n=1,3,5,\cdots)$$

We know the equation has the solution of form:

$$y_n = A_n \cos nt + B_n \sin nt$$

By substituting this into (*).

We find that

$$A_n = \frac{w^2}{n^2} - \frac{4}{n^4 \pi} \quad , \quad B_n = \frac{w^2}{n^2}$$

Since the equation is linear,

the solution is

$$y = y_0 + y_1 + y_3 + y_5 + \cdots$$

Problem 13

Represent $r(t)$ by a Fourier Series

Since $r(t)$ is odd, it can be represented by a Fourier sine series:

$$b_n = \frac{2}{\pi} \int_0^{\pi} r(t) \sin nt \, dt$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{t}{12} (\pi^2 - t^2) \sin nt \, dt$$

$$= \frac{1}{6\pi} \left[\int_0^{\pi} t \pi^2 \sin nt \, dt - \int_0^{\pi} t^3 \sin nt \, dt \right]$$

$$= \frac{1}{6\pi} \left[-\frac{\pi^3}{n} \cos n\pi + \frac{\pi^3}{n} \cos n\pi - \frac{6\pi}{n^3} \cos n\pi \right]$$

$$= -\frac{1}{n^3} \cos n\pi = (-1)^{n+1} \cdot \frac{1}{n^3}$$

So $r(t) = \sin t - \frac{1}{8} \sin 2t + \frac{1}{27} \sin 3t - \frac{1}{64} \sin 4t + \cdots$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \sin nt$$

Consider the equation

(*) $$y_n'' + c y' + y = -\frac{1}{n^3} \cos n\pi \cdot \sin nt.$$

We know the equation has the solution of form

$$y_n = A_n \cos nt + B_n \sin nt$$

By substituting this into (*).

We can find that :

$$A_n = \frac{(-1)^n \cdot c}{n^2 \cdot D} \qquad B_n = \frac{(-1)^n (n^2-1)}{n^3 \cdot D}$$

where $D = (n^2-1)^2 + (nc)^2.$

So the solution of the equation is

$$y = y_1 + y_2 + y_3 + \cdots$$

$$= \sum_{n=1}^{\infty} (A_n \cos nt + B_n \sin nt)$$

where $A_n = \dfrac{(-1)^n c}{n^2 D}$ $\qquad B_n = \dfrac{(-1)^n (n^2-1)}{n^3 D}$

$$D = (n^2-1)^2 + (nc)^2.$$

Problem 15 RLC - circuit is given by

$$L I'' + R I' + \frac{1}{C} I = E'(t)$$

For this problem, we have equation

$$10 I'' + 100 I' + 10^2 I = E'(t)$$

$$(*) \qquad I'' + 10 I' + 10 I = \frac{1}{10} E'(t) = F(t)$$

$$F(t) = \frac{1}{10} E'(t) = 20\pi^2 - 60 t^2 \qquad \text{if } -\pi < t < \pi$$

$$F(t + 2\pi) = F(t)$$

($F(t)$ has period 2π, since $E(t)$ has it).

Present $F(t)$ by a Fouries series.

Since $F(t)$ is even, it can be represented by a Fourier cosine series.

$$a_0 = \frac{1}{\pi} \int_0^{\pi} F(t)\, dt$$

$$= \frac{1}{\pi} \int_0^{\pi} (20\pi^2 - 60t^2)\, dt$$

$$= 0$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} F(t) \cos nt\, dt$$

$$= \frac{2}{\pi} \int_0^{\pi} (20\pi^2 - 60t^2) \cos nt\, dt$$

$$= \frac{40}{\pi} \left[\int_0^{\pi} \pi^2 \cos nt\, dt - \int_0^{\pi} 3t^2 \cos nt\, dt \right]$$

$$= \frac{40}{\pi} \left[0 - \frac{6\pi}{n^2} \cos n\pi \right]$$

$$= -\frac{240}{n^2} \cos n\pi$$

$$= (-1)^{n+1} \frac{240}{n^2}$$

So $$F(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{240}{n^2} \cos nt$$

$$= 240 \left[\cos t - \frac{1}{4} \cos 2t + \frac{1}{9} \cos 3t - \frac{1}{16} \cos 4t + \cdots \right].$$

Consider the equation

(*) $$I_n'' + 10 I_n' + 10 I_n = (-1)^{n+1} \frac{240}{n^2} \cos nt$$

We know the equation has the solution of form

$$I_n = A_n \cos nt + B_n \sin nt$$

By substituting this into (*).

we find that

$$A_n = (-1)^{n+1} \frac{240(100-n^2)}{n^2 D}, \quad B_n = (-1)^{n+1} \frac{2400}{n D}$$

where $D = (100-n^2)^2 + (10n)^2$

So the solution is

$$I = I_1 + I_2 + I_3 + \cdots$$