Section 11.5 #9 :

$$\begin{cases} u_t = c^2 u_{xx} \\ u_x(0,t) = 0 = u_x(L,t) \\ u(x,0) = f(x) \end{cases}$$

Let $u(x,t) = X(x)T(t)$

then $u_t = c^2 u_{xx}$ becomes : $T'X = c^2 T X''$

$$\Rightarrow \frac{T'}{c^2 T} = \frac{X''}{X} = k \quad \text{for some constant } k.$$

Notice that

$u_x(0,t) = X'(0)T(t) = 0 \quad \Rightarrow X'(0) = 0$

$u_x(L,t) = X'(L)T(t) = 0 \quad \Rightarrow X'(L) = 0$

$X'' = kX$ has different solutions depending on whether k is positive, zero, or negative:

if $k > 0$: let $k = p^2$. Then $X'' = p^2 X$ and $X(x) = c_1 e^{px} + c_2 e^{-px}$

$\Rightarrow X'(x) = c_1 p e^{px} - c_2 p e^{-px}$, so $X'(0) = c_1 - c_2 = 0$

and $\boxed{c_1 = c_2}$.

Then $X'(L) = c_1 p[e^{pL} - e^{-pL}] = 0 \Rightarrow \boxed{c_1 = 0}$

and the only solution is $X(x) = 0$, the trivial solution.

if $k = 0$: $X'' = 0 \quad \Rightarrow \quad X(x) = ax + b.$

Thus $X'(x) = a$, and $X'(0) = 0 = a$, $X'(L) = 0 = a$

so $a = 0$, and b is undetermined. $X(x) = b$.

Here, $T' = 0$ as well, which implies that $T(t) = c$ for some constant c. Thus $u(x,t) = X(x)T(t) = C_0$ is a solution for the case $k = 0$, if $k<0$: Let $k=-p^2$. Then $X''=-p^2X$

and $X(x)=c_1\cos px + c_2\sin px$

$\Rightarrow X'(x) = -c_1 p\sin px + c_2 p\cos px$

$X'(0) = c_2 = 0$

$X'(L) = -c_1 p\sin(pL) = 0 \Rightarrow pL = n\pi$

$\Rightarrow \boxed{p_n = \frac{n\pi}{L}}$ where n is an integer.

so $X_n(x) = c_n\cos\left(\frac{n\pi x}{L}\right)$

now: $T_n' = -p_n^2 T_n \Rightarrow T_n(t) = b_n e^{-p_n^2 t}$.

and so $u_n(x,t) = C_n e^{-p_n^2 t}\cos\left(\frac{n\pi x}{L}\right)$ with $p_n = \frac{n\pi}{L}$

and if we add together all of our solutions, we obtain

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} C_n e^{-p_n^2 t}\cos\left(\frac{n\pi x}{L}\right)$$

where the solution c_0 was obtained from the case $k=0$, and the rest from $k<0$.

To satisfy the initial condition $u(x,0)=f(x)$:

$u(x,0) = c_0 + \sum_{n=1}^{\infty} c_n\cos\left(\frac{n\pi x}{L}\right) = f(x)$ $\Rightarrow$ we have the c_ns are the F.S. coeffs for the even $2L$-periodic extension of $f(x)$.

so, $c_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$, $c_0 = \frac{1}{L}\int_0^L f(x)\,dx$.