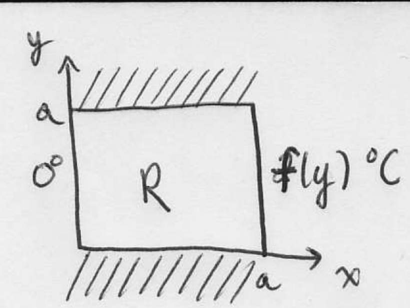
19.



$u(0,y) = 0$
$u(a,y) = f(y)$
$u_y(x,0) = 0 = u_y(x,a)$

$u_{xx} + u_{yy} = 0$

let $u(x,y) = X(x)Y(y)$

$\Rightarrow \quad X''Y + XY'' = 0$

$\Rightarrow \quad \dfrac{X''}{X} = -\dfrac{Y''}{Y} = k$ for some constant k.

note that $u_y(x,y) = X(x)Y'(y)$

$\Rightarrow u_y(x,0) = X(x)Y'(0) = 0 \quad \Rightarrow Y'(0) = 0$

$u_y(x,a) = X(x)Y'(a) = 0 \quad \Rightarrow Y'(a) = 0$

$u(0,y) = X(0)Y(y) = 0 \quad \Rightarrow X(0) = 0$

[$u(a,y) = X(a)Y(y) = f(y)$ inconclusive. $\rightarrow$ apply this last]

So we have: $\quad X'' - kX = 0 \quad , \quad X(0) = 0$

$Y'' + kY = 0 \quad , \quad Y'(0) = 0 = Y'(a)$

<u>if $k < 0$</u> $\Rightarrow k = -p^2 \Rightarrow Y'' - p^2 Y = 0 \Rightarrow Y(y) = c_1 e^{py} + c_2 e^{-py}$

$\Rightarrow Y'(y) = p(c_1 e^{py} - c_2 e^{-py})$

$Y'(0) = p(c_1 - c_2) = 0 \Rightarrow \boxed{c_1 = c_2}$

$Y'(a) = p(c_1 e^{pa} - c_1 e^{-pa}) = 0 \quad \Rightarrow c_1 = 0.$

So $c_1 = c_2 = 0$ and $Y(y) = 0 \Rightarrow$ only trivial solution in this case.

if $k=0$:

$$Y'' = 0 \Rightarrow Y = \alpha y + b$$
$$Y' = \alpha$$

$$\left.\begin{array}{l} Y'(0) = \alpha = 0 \\ Y'(a) = \alpha = 0 \end{array}\right\} \Rightarrow Y = b$$

but we also have:

$$X'' = 0 \quad \text{if } k=0$$
$$\Rightarrow X = \alpha x + c$$
$$X(0) = 0 \Rightarrow c = 0$$

so $X(x) = \alpha x$, and $u(x,y) = X(x)Y(y) = Cx$

but $u(a,y) = Ca = f(y)$ cannot hold unless $f(y)$ is constant!

so if $f(y) = \omega$ (constant)

then we have $Ca = \omega \Rightarrow C = \frac{\omega}{a}$

and $u(x,y) = \frac{\omega}{a} x$ solves the PDE.

Otherwise, if $f(y)$ is not constant, there is no solution in this case.

if $k>0$: let $k = p^2$:

$$X'' = p^2 X, \quad Y'' + p^2 Y = 0$$

$$\Rightarrow Y(y) = c_1 \cos py + c_2 \sin py$$
$$Y'(y) = -pc_1 \sin py + c_2 p \cos py$$

then $Y'(0) = c_2 p = 0 \Rightarrow c_2 = 0$

$Y'(a) = -p c_1 \sin(pa) = 0 \Rightarrow pa = n\pi$

$$\boxed{p_n = \frac{n\pi}{a}}$$

so $Y(y) = c_n \cos\left(\frac{n\pi y}{a}\right)$

Now to find $X(x)$: $p_n = \frac{n\pi}{a}$ and $X'' = p_n^2 X$

$$\Rightarrow X_n(x) = c_1 e^{p_n x} + c_2 e^{-p_n x}$$

$X_n(0) = c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

So $X_n(x) = c_n\left(e^{p_n x} - e^{-p_n x}\right)$

and $u(x,y) = \sum_{n=1}^{\infty} C_n\left(e^{p_n x} - e^{-p_n x}\right)\cos\left(\frac{n\pi y}{a}\right)$

$= \sum_{n=1}^{\infty} c_n \sinh(p_n x)\cos\left(\frac{n\pi y}{a}\right)$

We still need $u(a,y) = f(y)$:

$$u(a,y) = \sum_{n=1}^{\infty} c_n \sinh(n\pi)\cos\left(\frac{n\pi y}{a}\right) = f(y)$$

If $\int_0^L f(y)\,dy = 0$, then this is the fourier cosine series for the even $2a$-periodic extension of $f(y)$, and

$$c_n = \frac{2}{a\sinh(n\pi)}\int_0^a f(y)\cos\left(\frac{n\pi y}{a}\right)dy$$

If $\int_0^L f(y)\,dy \neq 0$, then there is no solution in this case...

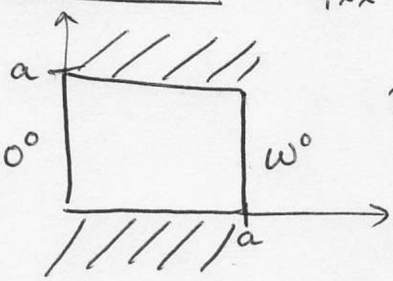
So, we only get a solution to this BVP if either $f(y)$ is constant OR $\int_0^L f(y)\,dy = 0$... OR, we can be clever & put these ideas together

So according to what we have done,

if we let
$$w = \frac{1}{a}\int_0^a f(y)\,dy \qquad \text{(notice } w \text{ is a constant!)}$$

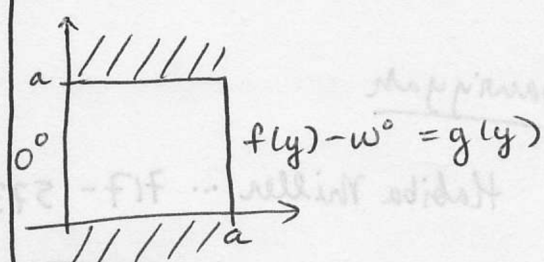
then we can solve Laplace's equation on:

Situation 1 : $u_{1xx} + u_{1yy} = 0$



$$u_1(x,y) = \frac{wx}{a}$$

Situation 2 : $u_{2xx} + u_{2yy} = 0$



$$u_2(x,y) = \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi y}{a}\right)\sinh\left(\frac{n\pi x}{a}\right)$$

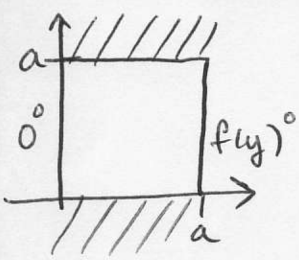
with
$$c_n = \frac{2}{a\sinh(n\pi)}\int_0^a [f(y) - w]\cos\left(\frac{n\pi y}{a}\right)dy$$

because
$$\int_0^a g(y) = \int_0^a f(y) - w^\circ = \int_0^a f(y) - \int_0^a \frac{1}{a}\int_0^a f(y)\,dy$$
$$= \int_0^a f(y)\,dy - \int_0^a f(y)\,dy = 0.$$

now, notice that

$$u(x,y) = u_1(x,y) + u_2(x,y)$$

solves our situation, because:



$$u(x,y) = \frac{wx}{a} + u_2(x,y)$$

$$u(0,y) = u_2(0,y) = 0^\circ$$
$$u(a,y) = w + u_2(a,y) = w + f(y) - w = f(y)$$
$$u_y(x,0) = u_{2y}(x,0) = 0$$
$$u_y(x,a) = u_{2y}(x,a) = 0$$
and $u_{xx} = u_{2xx}$, $u_{yy} = u_{2yy}$

$$\Rightarrow u_{xx} + u_{yy} = 0$$

So :
$$u(x,y) = \frac{wx}{a} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi y}{a}\right)\sinh\left(\frac{n\pi x}{a}\right)$$
solves the problem