#19:
$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = 0 = u_x(L,t) \\ u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases}$$

Solution: Let $u(x,t) = X(x)T(t)$

then $u_{tt} = c^2 u_{xx}$ becomes $T''X = c^2 X'' T$

and $\dfrac{T''}{c^2 T} = \dfrac{X''}{X} = k$ (for some constant k)

if $k < 0$: we can write $k = -p^2$ for some p.

then: $X'' = kX = -p^2 X$

notice that $u(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0$

and $u_x(L,t) = X'(L)T(t) = 0 \Rightarrow X'(L) = 0$

$X'' = -p^2 X \Rightarrow X(x) = c_1 \cos px + c_2 \sin px$

$X(0) = c_1 = 0$

$X'(L) = c_2 p \cos pL = 0 \Rightarrow \cos(\underbrace{pL}) = 0$, so $pL = \dfrac{(2n+1)\pi}{2}$

this must be an odd multiple of $\pi/2$.

so $\boxed{p_n = \dfrac{(2n+1)\pi}{2L}}$ and $X_n(x) = C_n \sin\left(\dfrac{(2n+1)\pi x}{2L}\right)$

$\dfrac{T_n''}{c^2 T_n} = -p_n^2 \Rightarrow T_n(t) = b_n \cos(p_n c t) + b_n^* \sin(p_n c t)$

Since $u_t(x,0) = X(x)T'(0) = 0 \Rightarrow T'(0) = 0$

so $T_n'(0) = b_n^* p_n c = 0 \Rightarrow b_n^* = 0$, and $\boxed{u_n(x,t) = C_n \cos(p_n c t)\sin(p_n x)}$

if $k=0$: $\quad X = ax+b$

$$\left.\begin{aligned} X(0) &= b = 0 \\ X'(L) &= a = 0 \end{aligned}\right\} \Rightarrow \text{only trivial solution is possible}$$

if $k>0$: $\quad X(x) = c_1 e^{px} + c_2 e^{-px}$

$(k=p^2)$ $\quad X(0) = c_1 + c_2 = 0 \quad \Rightarrow c_1 = -c_2$

$$X'(x) = p\left(c_1 e^{px} + c_1 e^{-px}\right)$$

$$\Rightarrow X'(L) = c_1 p\left(e^{pL} + e^{-pL}\right) = 0$$

$$\Rightarrow c_1 = 0$$

so again only the trival solution is possible!

thus:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(p_n x)\cos(p_n ct) = \sum_{n=1}^{\infty} u_n(x,t)$$

and $\displaystyle A_n = \frac{2}{L}\int_0^L f(x)\sin(p_n x)\,dx$

where $\displaystyle p_n = \frac{(2n+1)\pi}{2L}$

solves the problem, since $\displaystyle \sum_{n=1}^{\infty} c_n \sin(p_n x) = u(x,0) = f(x)$