Section 11.3 #10 :

$$\begin{cases} u_{tt} = u_{xx}, \quad u(0,t) = 0 = u(\pi,t) \\ u(x,0) = 0 \\ u_t(x,0) = \begin{cases} 0.01x & 0 \le x \le \frac{1}{2}\pi \\ 0.01(\pi - x) & \pi/2 \le x \le \pi \end{cases} = f(x) \end{cases}$$

solution: we know that the general solution is:

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos\left(\tfrac{n\pi t}{L}\right) + B_n^* \sin\left(\tfrac{n\pi t}{L}\right) \right) \sin\left(\tfrac{n\pi x}{L}\right)$$

for the wave equation with $c^2 = 1$ and $u(0,t) = 0 = u(\pi,t)$ (zero boundary conditions). So, let's apply the initial conditions:

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\tfrac{n\pi x}{L}\right) = 0 \Rightarrow B_n = 0 \text{ for all } n.$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \left(-B_n \frac{n\pi}{L} \sin\left(\tfrac{n\pi t}{L}\right) + \frac{n\pi}{L} B_n^* \cos\left(\tfrac{n\pi t}{L}\right) \right) \sin\left(\tfrac{n\pi x}{L}\right)$$

$$\Rightarrow u_t(x,0) = \sum_{n=1}^{\infty} \underbrace{\frac{n\pi}{L} B_n^*}_{c_n} \sin\left(\tfrac{n\pi x}{L}\right) = f(x)$$

thus we have a fourier series expansion for the odd 2π-periodic extension of $f(x)$ and since $L = \pi$:

$$c_n = \frac{n\pi}{\pi} B_n^* = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\,dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} 0.01x \sin(nx)\,dx + \int_{\pi/2}^{\pi} (0.01)(\pi - x) \sin(nx)\,dx \right]$$

$$= \frac{.02}{\pi} \left[-\frac{x}{n} \cos(nx) \Big|_0^{\pi/2} + \frac{1}{n} \int_0^{\pi/2} \cos(nx)\,dx - \frac{(\pi - x)}{n} \cos(nx) \Big|_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} \frac{1}{n} \cos(nx)\,dx \right]$$

$$= \frac{.02}{\pi} \left[\cancel{-\frac{\pi}{2n} \cos\left(\tfrac{n\pi}{2}\right)} + \frac{1}{n^2} \sin\left(\tfrac{n\pi}{2}\right) + \cancel{\frac{\pi}{2n} \cos\left(\tfrac{n\pi}{2}\right)} - \frac{1}{n^2} \left(\underset{0}{\sin(n\pi)} - \sin\left(\tfrac{n\pi}{2}\right) \right) \right] = \frac{.04}{\pi n^2} \sin\left(\tfrac{n\pi}{2}\right)$$

so $\quad c_n = \frac{n\pi}{\pi} B_n^* = n B_n^* = \frac{.04}{\pi n^2} \sin\left(\frac{n\pi}{2}\right)$

$$\Rightarrow \quad B_n^* = \frac{.04}{\pi n^3} \sin\left(\frac{n\pi}{2}\right)$$

and

$$u(x,t) = \sum_{n=1}^{\infty} \frac{.04}{\pi n^3} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi t}{\pi}\right) \sin\left(\frac{n\pi x}{\pi}\right)$$

$$= \sum_{n=1}^{\infty} \frac{.04}{\pi n^3} \sin\left(\frac{n\pi}{2}\right) \sin(nt) \sin(nx).$$