

Section 11.3 #10:

$$\begin{cases} u_{tt} = u_{xx}, & u(0,t) = 0 = u(\pi, t) \\ u(x,0) = \begin{cases} 0.01x & 0 \leq x \leq \frac{1}{2}\pi \\ 0.01(\pi-x) & \frac{1}{2}\pi \leq x \leq \pi \end{cases} = f(x) \end{cases}$$

Solution: we know that the general solution is:

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos\left(\frac{n\pi t}{L}\right) + B_n^* \sin\left(\frac{n\pi t}{L}\right) \right) \sin\left(\frac{n\pi x}{L}\right)$$

(or the wave equation with $c^2 = 1$ and $u(0,t) = 0 = u(\pi,t)$ (zero boundary conditions). So, let's apply the initial conditions:

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = 0 \Rightarrow B_n = 0 \text{ for all } n.$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \left(-B_n \frac{n\pi}{L} \sin\left(\frac{n\pi t}{L}\right) + \frac{n\pi}{L} B_n^* \cos\left(\frac{n\pi t}{L}\right) \right) \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow u_t(x,0) = \sum_{n=1}^{\infty} \underbrace{\frac{n\pi}{L} B_n^*}_{c_n} \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

thus we have a Fourier series expansion for the odd 2π -periodic extension of $f(x)$ and since $L=\pi$:

$$\begin{aligned} c_n &= \frac{n\pi}{\pi} B_n^* = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx) dx \\ &= \frac{2}{\pi} \left[\int_0^{\frac{\pi}{2}} 0.01x \sin(nx) dx + \int_{\frac{\pi}{2}}^\pi (0.01)(\pi-x) \sin(nx) dx \right] \\ &= \frac{0.2}{\pi} \left[-\frac{x}{n} \cos(nx) \Big|_0^{\frac{\pi}{2}} + \frac{1}{n} \int_0^{\frac{\pi}{2}} \cos(nx) dx - \frac{(\pi-x)}{n} \cos(nx) \Big|_{\frac{\pi}{2}}^\pi - \frac{1}{n} \int_{\frac{\pi}{2}}^\pi \cos(nx) dx \right] \\ &= \frac{0.2}{\pi} \left[-\frac{\pi}{2n} \cos\left(\frac{n\pi}{2}\right) + \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right) + \frac{\pi}{2n} \cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2} (\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right)) \right] = \frac{0.04}{\pi n^2} \sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

$$\text{So } C_n = \frac{n\pi}{\pi} B_n^* = n B_n^* = \frac{0.04}{\pi n^2} \sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow B_n^* = \frac{0.04}{\pi n^3} \sin\left(\frac{n\pi}{2}\right)$$

and

$$u(x,t) = \sum_{n=1}^{\infty} \frac{0.04}{\pi n^3} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi t}{\pi}\right) \sin\left(\frac{n\pi x}{\pi}\right)$$

$$= \sum_{n=1}^{\infty} \frac{0.04}{\pi n^3} \sin\left(\frac{n\pi}{2}\right) \sin(nt) \sin(nx).$$

$$= \frac{\pi}{T} \left[-\frac{e^{i\pi}}{3} \left(e^{-i\pi/2} - e^{i\pi/2} \right) + \frac{1}{3} e^{i\pi} \left(e^{-i\pi/2} - e^{i\pi/2} \right) \right]$$

$$= \frac{\pi}{T} \left[-\frac{e^{i\pi}}{3} \left(e^{-i\pi/2} - e^{i\pi/2} \right) - \frac{1}{3} e^{i\pi} \left(e^{-i\pi/2} - e^{i\pi/2} \right) \right]$$

$$= \frac{\pi}{T} \left[\frac{e^{i\pi}}{3} e^{-i\pi/2} \left[1 - \frac{e^{i\pi}}{3} e^{-i\pi/2} \right] - \frac{1}{3} e^{i\pi} e^{-i\pi/2} \left[1 - \frac{e^{i\pi}}{3} e^{-i\pi/2} \right] \right]$$

$$\Rightarrow C^* = \frac{\pi}{T} \left[r(t) e^{-i\pi/2} \right] = \frac{\pi}{T} \left[\left[\begin{array}{l} 1 \\ -\frac{1}{3} e^{i\pi} e^{-i\pi/2} \end{array} \right] + \left[\begin{array}{l} e^{-i\pi} e^{-i\pi/2} \\ -\frac{1}{3} e^{i\pi} e^{-i\pi/2} \end{array} \right] \right]$$

$$r(t) = \begin{cases} 3 & R^2 < t < 1 \\ 1 & -R^2 < t < R^2 \\ 3 & -1 < t < R^2 \end{cases}$$

$$C^* = \frac{\pi r}{T} \left[\begin{array}{l} r(t) e^{-i\pi/2} \\ -\frac{1}{3} e^{i\pi} e^{-i\pi/2} \end{array} \right]$$

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