#4. $u_x + 2x u_t = 2x \qquad u(x,0) = 1, \; u(0,t) = 1$

Applying the L.T. in t:

$$\mathcal{L}\{u_x\} + 2x\,\mathcal{L}\{u_t\} = 2x\,\mathcal{L}\{1\}$$

we assume that:

$$\mathcal{L}\{u_x\} = \int_0^\infty e^{-st} u_x(x,t)\,dt = \int_0^\infty e^{-st} \frac{d}{dx} u(x,t)\,dt$$

$$= \frac{d}{dx}\int_0^\infty e^{-st} u(x,t)\,dt = \frac{d}{dx}\mathcal{L}\{u\} = \mathcal{U}_x$$

$$\Rightarrow \mathcal{U}_x + 2x\left(s\,\mathcal{U} - u(x,0)\right) = 2x \cdot \frac{1}{s}$$

$$\mathcal{U}_x + 2xs\,\mathcal{U} = \frac{2x}{s} + 2x$$

homog: $\mathcal{U}_x + 2xs\,\mathcal{U} = 0 \Rightarrow \dfrac{\mathcal{U}_x}{\mathcal{U}} = -2xs$

$$\Rightarrow \int \frac{\mathcal{U}_x}{\mathcal{U}}\,dx = \int -2xs\,dx \Rightarrow \ln \mathcal{U} = -\frac{2x^2 s}{2} + C(s)$$

$$\Rightarrow \mathcal{U}_h = Ce^{-x^2 s}$$

inhomog: try $\mathcal{U} = \alpha x + \beta \Rightarrow \mathcal{U}_x = \alpha$

so: $\alpha + 2xs(\alpha x + \beta) = 2x\left(1 + \frac{1}{s}\right)$

$$\Rightarrow \alpha = 0, \quad 2s\beta = 2\left(1 + \frac{1}{s}\right) \Rightarrow \beta = \frac{1}{s} + \frac{1}{s^2}$$

and $\mathcal{U}_p = \frac{1}{s} + \frac{1}{s^2}$

$$\mathcal{U} = \mathcal{U}_h + \mathcal{U}_p = C(s)e^{-x^2 s} + \frac{1}{s} + \frac{1}{s^2} \Rightarrow u(x,t) = C(t - x^2)u(t - x^2) + 1 + t$$

we have

$u(0,t) = 1$, so $u(0,t) = C(t) + 1 + t = 1 \Rightarrow C(t) = -t$

$\Rightarrow \quad u(x,t) = -(t-x^2)u(t-x^2) + 1 + t$

#5) $x u_x + u_t = xt \qquad u(x,0)=0 \; , \; u(0,t)=0 \quad (x \geq 0, t \geq 0)$

applying L.T. in t:

$$x\,\mathcal{U}_x + s\,\mathcal{U} = \frac{x}{s^2}$$

homog:

$$\frac{\mathcal{U}_x}{\mathcal{U}} = -\frac{s}{x} \Rightarrow \ln \mathcal{U} = -s\ln x + C(s)$$

$$\Rightarrow \mathcal{U}_h = e^{-s\ln x + c} = Ce^{\ln x^{-s}} = Cx^{-s}$$

inhomog:

$$x\,\mathcal{U}_x + s\,\mathcal{U} = \frac{1}{s^2}x \quad , \quad \text{try } \mathcal{U} = \alpha x + \beta$$

$$x\alpha + s(\alpha x + \beta) = \frac{1}{s^2}x \Rightarrow \alpha + s\alpha = \frac{1}{s^2} \quad , \quad \beta = 0$$

$$\alpha = \frac{1}{s^2(s+1)} \Rightarrow \mathcal{U}_p = \frac{x}{s^2(s+1)}$$

so $\mathcal{U} = \mathcal{U}_h + \mathcal{U}_p = C(s)\,x^{-s} + \dfrac{x}{s^2(s+1)}$

since $u(0,t) = 0 \Rightarrow \mathcal{U}(0,s) = \mathcal{L}\{u(0,t)\} = 0$

$$\mathcal{U}(0,s) = \frac{C(s)}{\underbrace{0^s}_{\text{undefined!}}} + \frac{0}{s^2(s+1)} \Rightarrow C(s) = 0 \text{ needs to be true.}$$

so $\mathcal{U}(x,s) = \dfrac{x}{s^2(s+1)} \Rightarrow u(x,t) = x\,\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s+1)}\right\} = x\displaystyle\int_0^t\int_0^z e^{-\tau}\,d\tau\,dz$

$$= x\left(\int_0^t -e^{-z} + 1\,dz\right) = x\left(e^{-t} + t - 1\right)$$

$$\boxed{u(x,t) = x\left(e^{-t} + t - 1\right)}$$

7. <u>model:</u>

<u>heat eq'n:</u>

$$w_t = c^2 w_{xx}$$

<u>boundary conditions:</u>

$$\begin{cases} \lim\limits_{x\to\infty} w(x,t) = 0 \\ w(0,t) = f(t) \end{cases}$$

<u>initial condition:</u>

$$w(x,0) = 0$$

taking the L.T.:

$$sW = c^2 W_{xx} \Rightarrow W_{xx} - \frac{s}{c^2} W = 0$$

try $W = e^{\lambda x} \Rightarrow \lambda = \pm \frac{\sqrt{s}}{c}$

$$W(x,s) = C_1(s) e^{\sqrt{s}/c\, x} + C_2(s) e^{-\sqrt{s}/c\, x}$$

since $\lim\limits_{x\to\infty} W(x,s) = \mathcal{L}\left\{\lim\limits_{x\to\infty} w(x,t)\right\} = \mathcal{L}\{0\} = 0$

we see that $C_1(s) = 0$ must be true b/c

$$\lim_{x\to\infty} e^{\sqrt{s}/c\, x} = \infty$$

so $W(x,s) = C_2(s) e^{-\sqrt{s}/c\, x}$

Now: $W(0,s) = \mathcal{L}\{w(0,t)\} = \mathcal{L}\{f(t)\} = F(s)$

$$\Rightarrow W(0,s) = C_2(s) = F(s)$$

$$\Rightarrow W(x,s) = F(s) e^{-\sqrt{s}/c\, x}$$

#8) From #7, we have

$$W(x,s) = F(s)e^{-\sqrt{s}/c\, x}$$

$$\Rightarrow w(x,t) = \mathcal{L}^{-1}\left\{ F(s)e^{-\sqrt{s}/c\, x}\right\}$$

we have two facts (from the tables at the end of Ch5)

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g \quad \text{where } \mathcal{L}\{f\}=F\ ,\ \mathcal{L}\{g\}=G$$

$$\mathcal{L}^{-1}\left\{ e^{-k\sqrt{s}}\right\} = \frac{k}{2\sqrt{\pi t^3}}\, e^{-k^2/4t}$$

$$\Rightarrow w(x,t) = f(t) * \left(\frac{x/c}{2\sqrt{\pi t^3}}\, e^{-x^2/4tc^2}\right)$$

$$= \int_0^t f(t-\tau)\,\frac{x/c}{2\sqrt{\pi \tau^3}}\, e^{-x^2/4\tau c^2}\, d\tau$$

$$= \frac{x}{2c\sqrt{\pi}} \int_0^t f(t-\tau)\, \tau^{3/2}\, e^{-x^2/4\tau c^2}\, d\tau$$