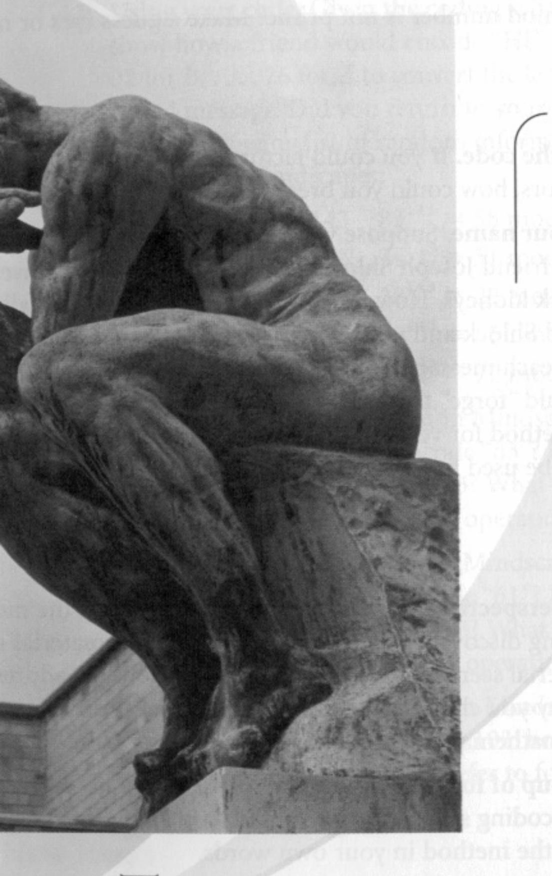
The Irrational Side of Numbers

Are There Numbers Beyond Fractions?



> God made integers, all else is the work of humankind.
>
> LEOPOLD KRONECKER

The natural numbers are the first and most natural measures of quantity; however, suppose we have more than one of something but less than two? Clearly we need fractional quantities to make such measurements. Fractions let us measure any quantity to any desired degree of precision. In principle, we could measure a length to within one-millionth of an inch. But are there quantities that even the most precise fraction cannot measure exactly? Specifically, is every number a fraction? That question challenged the ancient Greeks and eventually opened their minds to a totally new and surprising realization about the notion of "number." This discovery of the Greeks, which we will soon discover for ourselves, is another powerful illustration of our major theme: By asking clear questions and examining the familiar in a careful and logical manner, we uncover hidden richness.

Is every number a fraction? To answer this question we make an assumption and follow the consequences of doing so. Letting logic lead, we suppose that every length could be measured exactly as a fraction, and then we see what other results we would be compelled, by logic, to accept. Exploring the logical consequences of an assumption is a valuable way to determine whether the assumption is reasonable.

A Rational Mindset

The ancient Greeks, and probably people before them, devised a reasonable method of measuring parts of things. If we take an object and break it into 10 equal pieces and take 9 of them, then we have measured nine-tenths (9/10). Those who sell gasoline at the corner store have learned this lesson well—every gas price ends in 9/10 of a cent. This clever ploy allows the neighborhood convenience store to milk us for a smidgen extra on each gallon without our really noticing.

Theoretically we could take an object, break it into a million pieces, and take 375,687 of them and get 375,687 millionths (375,687/1,000,000). So by taking a large enough number of pieces, we can measure parts of things to any degree of accuracy we want; unfortunately, even then we may not be correct. But we are getting a bit ahead of ourselves. The Greeks thought that the natural numbers were natural gifts from the gods. Ratios of those number essences together with their negatives and zero produce the rational numbers. A *rational number,* therefore, is a number that can be written as a fraction a/b or $-a/b$ where a and b are natural numbers or where $a = 0$. Some examples of rational numbers are 1/2, 22/7, 109/51, −35/219, 15 (15/1), and 0.

To get accustomed to this idea, find a rational number between 1 and 2. Now find a rational number between 1001/1003 and 1002/1003. Why is there always a rational number between any two other rational numbers?

Notice that, even if two rational numbers are very close together, we can always find many (in fact, infinitely many) rational numbers between them. Since we can cut things up into as many equal-size pieces as we wish, it seems reasonable to conjecture that every number is rational, which the early Greeks believed. Given common observations and life experiences, this idea seems both natural and rational (excuse the pun). In fact, the atomic theory of matter and quantum mechanics suggests that matter has a limit to its divisibility, and, hence, for physical objects, there may be a specific number of indivisible units that make up the object. So if we break an object arbitrarily in two and wish to measure how big each piece is, we would count the number of particles in one piece and divide that number by the total number of particles in the original object to see what fraction of the object we have.

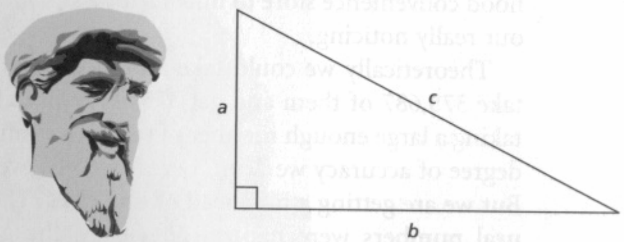
Of course, mathematics is not constrained by mere physical reality. Physical reality is just the starting point for mathematics.

The Pythagoreans' Secret Society and the Square Root of 2

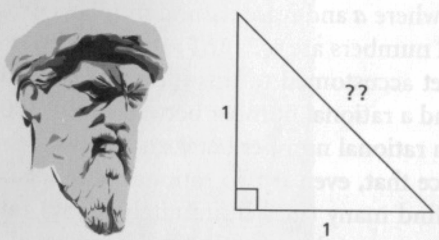
Let's now assume, as the Greeks did, the reasonable hypothesis that all numbers are rational numbers and see where that assumption leads us. Unfortunately, it leads us into some deep trouble. Along the way, however, we will learn that an effective method for discovering new ideas and truths is to explore the consequences of assumptions.

Pythagoras (580 BCE–500 BCE) and his followers formed a school devoted to discovering great ideas, many of which were mathematical. The Pythagorean School was a secret society. They developed important mathematical concepts and kept them to themselves.

The Pythagorean Theorem, which we will physically see and feel in Chapter 4: "Geometric Gems," tells us that in a right triangle the square of the hypotenuse equals the sum of the squares of the lengths of the two shorter sides; that is, $a^2 + b^2 = c^2$ where



After the Pythagoreans discovered this relationship, they considered a triangle with $a = b = 1$:



and wondered what the length of the hypotenuse was. If they called the length H, they knew that $1^2 + 1^2 = H^2$. So $H^2 = 2$, which is to say that H is the square root of 2, denoted as $H = \sqrt{2}$.

Assume It's Rational

The Pythagoreans believed that all numbers are rational, so the number H, the square root of 2, would have to be equal to some fraction, say a/b, where both a and b are natural numbers. Let's see where that assumption led the Pythagoreans and where it leads us.

The first thing we can do with our assumption that H equals a/b is to cancel any common factors in a and b until we are left with an equivalent rational number with the top number and the bottom number having no common factors. For example,

$$\frac{60}{45} = \frac{2^2 \times 3 \times 5}{3^2 \times 5} = \frac{2^2}{3} = \frac{4}{3}.$$

Notice how we factored every number into primes and then canceled common primes. So the rational 60/45 is equal to 4/3. Notice that 3 and 4 have no common factors.

NO COMMON FACTORS

Let's return to H, the square root of 2. We are assuming that H is a rational number a/b, and by following the cancellation process, we find that we can write H as c/d, where c and d share no common factor other than 1. In particular (and this observation is important) if 2 divides evenly into either of the numbers c or d, then 2 will not divide evenly into the other one. That is, $H = c/d$ and **not both** c and d are even numbers (since otherwise they would have a common factor of 2).

On the one hand, $H = \sqrt{2}$, and on the other hand, $H = c/d$. So $\sqrt{2} = c/d$. To simplify that equation, let's square both sides of $\sqrt{2} = c/d$. Doing so would produce

$$2 = \frac{c^2}{d^2}.$$

Since natural numbers are easier to visualize than fractions, let's multiply each side by d^2:

$$2d^2 = c^2.$$

HERE'S LOOKING AT *c*

Now let's see where this equation leads. What kind of number is c^2? Well, we see it equals $2d^2$, so 2 is a factor of c^2 and therefore c^2 must be even. But if c^2 is even, c itself is even. (Why? Well, we know 2 is a prime number, and if 2 divides evenly into $c \times c$ then 2 must divide evenly into just c alone.) So c is an EVEN number. Since c is an even number, then it must equal 2 times some other number, say $c = 2n$ (since that's what it means to be even, after all). If we substitute $2n$ for c in the equality above, we see that

$$2d^2 = c^2 = (2n)^2 = (2n)(2n) = 4n^2.$$

Looking at this equation, we have this unstoppable desire to divide both sides by 2, which leads to $d^2 = 2n^2$, which, in turn, leads to trouble.

A TROUBLING "*d*" TOUR

What kind of number is d^2? It must be even. Thus d itself must also be EVEN. Remember that we started by assuming that the square root of 2 was equal to a rational number a/b. We made some legal deductions from that assumption, namely, after canceling, $a/b = c/d$ where c and d are not both even. Then we deduced that c and d must both be even—directly contradicting ourselves. So this situation is impossible. Therefore, whatever we assumed must be false.

But what did we assume? A rereading of our argument shows that we assumed only one thing, namely that $\sqrt{2}$ is a rational number. Since our assumption led us to a contradiction, our assumption must be false. So, $\sqrt{2}$ is not a rational number; that is, $\sqrt{2}$ is not equal to the ratio of two integers. A number that is not rational is called *irrational.* The observation about the square root of 2 is so important that we highlight it here:

> **Square Root of 2 Is Irrational.**
> *The square root of 2 is an irrational number.*

MAKING IT YOUR OWN

You can make this argument your own by working through the ideas and explaining them to a friend. The argument is beautiful and elegant, because a powerful and surprising result arises from some basic ideas creatively strung together. This proof appears in Euclid's book entitled *Elements* and is usually attributed to Euclid (ca. 300 BCE). However, evidence exists that Aristotle (384 BCE–322 BCE) also knew about this argument. Of course, we'll never know for sure who first discovered this counterintuitive revelation together with this elegant line of reasoning. Among most mathematicians, the proofs that there are infinitely many prime numbers and that the square root of 2 is irrational are considered to be among the most beautiful arguments in the field.

ACCEPTING REALITY

We may not like the idea that an entirely new kind of number must exist—numbers that are not rational; however, there is no use fighting it. No matter how much our previous worldview led us to believe that all numbers are expressible as the ratio of two integers, it just isn't so. We must accept the proven truth, embrace a new worldview, and explore the reality of numbers as they are. Once we prove something, we must add it to our list of truths and move on.

Of course, we are modern people, and we are unlikely to find the existence of irrational numbers a challenge to our philosophical biases about how the world is organized. But for the ancient Greeks, the ideas of proportion and ratios of whole numbers played a more central role in their understanding of reality. For them, the notion of irrational numbers significantly challenged their ideas about reality.

The Pythagoreans reacted strongly to the disturbing discovery of irrational numbers, and they kept the idea secret. When one of their members was caught revealing the secret, he was taken on a boat ride and thrown overboard. Think we're joking? Proclus around the 5th century CE gave a brief account:

"It is well known that the man who first made public the theory of irrationals perished in a shipwreck in order that the inexpressible and unimaginable should ever remain veiled. And so the guilty man, who fortuitously touched on and revealed this aspect of living things, was taken to the place where he began and there is forever beaten by the waves." In contrast, students today who divulge the mysteries of the irrational numbers and other scientific phenomena find themselves scooped up by major corporations offering impressive salaries.

BEYOND THE SQUARE ROOT OF 2

Once we discover an important idea, we should use it to deduce new or more general consequences.

Mathematicians are extremely frugal when it comes to ideas: Once they have one, they try to recycle and reuse it. By pushing an idea to its limits, we often uncover more than we first expected.

To illustrate this lesson, let's see if we can adapt the ideas used in the proof that the square root of 2 is irrational to show that the square root of 3 is irrational. If you try to work through the ideas on your own before reading on, you will gain a much deeper understanding of the ideas at work.

SQUARE ROOT OF 3

Let's assume that $\sqrt{3}$ is a rational number a/b, where a and b are natural numbers. By following the cancellation process, we find that we can write $\sqrt{3}$ as c/d, where c and d share no common factor other than 1. In particular (and this observation is important) if 3 divides evenly into either of the numbers c or d, then 3 will not divide evenly into the other one. That is, $\sqrt{3} = c/d$ and **not both** c and d have the factor of 3 (since otherwise we could cancel). If we now square both sides of the equation $\sqrt{3} = c/d$, we see that

$$3 = \frac{c^2}{d^2}.$$

If we multiply each side by d^2 to get integers on both sides of the equation, we have

$$3d^2 = c^2.$$

Now let's see what this equation means. We see that c^2 equals $3d^2$, so 3 is a factor of c^2. But if c^2 has a factor of 3, then c itself must have a factor of 3. (Why? Well, we know 3 is a prime number, and if 3 divides evenly into $c \times c$, then 3 must divide evenly into just c alone.) Since c has a factor of 3, it must equal 3 times some other integer, say $c = 3n$. If we substitute $3n$ for c in the preceding equality, we see that

$$3d^2 = (3n)^2 = (3n)(3n) = 9n^2.$$

We can now divide both sides by 3 and get $d^2 = 3n^2$. This last equation shows that d^2 must have a factor of 3 in it, and therefore d must have a factor of 3 as well. Thus we see that c and d share a common factor of 3. But we selected c and d so that they have no common factor greater than 1. This is a contradiction, and, therefore, this situation is impossible—hence our assumption must be false. So, $\sqrt{3}$ is not a rational number—it is irrational. Notice how this argument parallels our first one.

OTHER IRRATIONALS

In fact, we can use this method to show that the $\sqrt{6}$ and many other examples are irrational. We invite you to try these in the Mindscapes.

With a bit of care, we can extend this idea even further and show that $\sqrt{2} + \sqrt{3}$ is irrational. Again, let's assume that $\sqrt{2} + \sqrt{3}$ is actually a rational number, say a/b. If we square both sides, $(\sqrt{2} + \sqrt{3})^2 = (a/b)^2$, we have to be a bit careful expanding the left side. We do it here:

$$\begin{aligned}(\sqrt{2} + \sqrt{3})^2 &= (\sqrt{2} + \sqrt{3})(\sqrt{2} + \sqrt{3}) \\ &= (\sqrt{2} + \sqrt{3})\sqrt{2} + (\sqrt{2} + \sqrt{3})\sqrt{3} \\ &= \sqrt{2}\sqrt{2} + \sqrt{3}\sqrt{2} + \sqrt{2}\sqrt{3} + \sqrt{3}\sqrt{3} \\ &= 2 + \sqrt{6} + \sqrt{6} + 3 \\ &= 5 + 2\sqrt{6}.\end{aligned}$$

So we see that $5 + 2\sqrt{6} = a^2/b^2$, which means that $\sqrt{6} = (a^2 - 5b^2)/2b^2$. But the number on the right side is a fraction, so that means that $\sqrt{6}$ is a rational number. However, you will show in the Mindscapes that $\sqrt{6}$ is irrational. Therefore, we have reached another contradiction. Our first assumption must have been false, and we conclude that $\sqrt{2} + \sqrt{3}$ is irrational.

IRRATIONAL POWER

If we whittle the idea we are using down to its core, we can use it to prove that other more exotic numbers are irrational. Suppose $3^A = 9$. What would A equal? If $3^C = 27$, what would C equal? These questions are not too hard: $A = 2$, and $C = 3$. But suppose B is the number such that $3^B = 10$. What is B? We know from the previous two questions that B is bigger than 2 and smaller than 3, but there is no use trying to figure out exactly what decimal number it equals, because we can't: It's an irrational number. Why? Well, suppose that B were a rational number, say u/v, where both u and v are natural numbers. Then $3^{u/v} = 10$. If we raise both sides to the vth power, then the v's cancel out in the power on the left side:

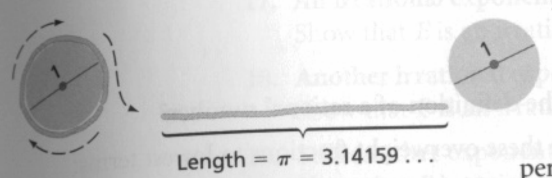
$$(3^{u/v})^v = 3^u = 10^v.$$

Since u and v are each at least 1, then 3 must divide evenly into 10^u, which is absurd: The only prime numbers that divide evenly into 10^v are 2's and 5's. This contradiction means that our assumption was false, so B must be an irrational number.

The number B is called a *logarithm*. If $3^B = 10$, then we would say that B is the logarithm of 10 in base 3. Using a calculator we can estimate B and see that $B = 2.09590327428938460\ldots$.

π

Our method to show certain numbers are irrational does not work for more exotic numbers. The circumference of a circle having diameter 1 is equal to the famous number π (pi). Although the rational number 22/7 is almost equal to π, it is not exactly equal to π: $22/7 = 3.142857142857\ldots$, while $\pi = 3.141592\ldots$. The Greeks and subsequent mathematicians studied π intensely.



In 1650 BCE, Egyptians estimated that $\pi \approx 256/81$, and roughly 500 years later, mathematicians in India approximated π as 62,832/20,000, which is incredibly close to π. It was not until 1761, however, that someone proved that, in fact, π is irrational. The first person to prove this important fact was Johann Lambert, and he used techniques from calculus. As an amusing postscript to Lambert's result and to the earlier works of Greek, Egyptian, and Indian mathematicians, we note that some ground was lost in 1897: The Indiana State Legislature considered a bill to declare π equal to 4, which was "offered as a contribution to education to be used only by the State of Indiana free of cost . . . ." Fortunately, the legislature did not pass the bill.

STILL UNKNOWN

In general, it is difficult to determine if numbers are rational or irrational. As a modest illustration, nobody knows if any of the numbers on the following list are irrational—it is possible (but not likely) that some are actually rational: 2^π, π^π, $\pi^{\sqrt{2}}$. Don't they all "look" irrational? Yes, but no one knows how to prove it for sure.

We now see that numbers come in two flavors: rational and irrational. The collection of all these numbers—rational and irrational—form the real numbers, which leads us to our final journey through the notion of number.

a look back

Beyond the world of natural numbers are the rational numbers, fractions. But some numbers are irrational—not rational. We can show that $\sqrt{2}$ is irrational by assuming the contrary. If $\sqrt{2}$ were rational, then it would be equal to a fraction written in lowest terms. That assumption implies that both the numerator and the denominator would have to have a factor of 2, which would contradict the fact that the fraction was in lowest terms. Thus $\sqrt{2}$ is not a fraction. This strategy can be used to demonstrate that other numbers are irrational. This reasoning allowed us to move from the comfortable world of natural numbers and their ratios to the real world of irrationality.

An effective strategy for analyzing life is to make an assumption and see what consequences follow logically. If a logical consequence is a contradiction, then the assumption must be wrong.

Follow assumptions to their logical conclusions.

32. **One-fourth root.** Show that the fourth root of 5, $\sqrt[4]{5}$, is irrational.

33. **Irrational sums (S).** Does an irrational number plus an irrational number equal an irrational number? If so, show why. If not, give some counter-examples.

34. **Irrational products.** Does an irrational number multiplied by an irrational number equal an irrational number? If so, show why. If not, give some counterexamples.

35. **Irrational plus rational.** Does an irrational number plus a rational number equal an irrational number? If so, show why. If not, give some counter-examples.

Challenges

36. **$\sqrt{p}$.** Show that for any prime number p, $\sqrt{p}$ is an irrational number.

37. **$\sqrt{pq}$.** Show that, for any two different prime numbers p and q, $\sqrt{pq}$ is an irrational number.

38. **$\sqrt{p} + \sqrt{q}$.** Show that, for any prime numbers p and q, $\sqrt{p} + \sqrt{q}$ is an irrational number.

39. **$\sqrt{4}$.** The square root of 4 is equal to 2, which is a rational number. Carefully modify the argument for showing that $\sqrt{2}$ is irrational to try to show that $\sqrt{4}$ is irrational. Where and why does the argument break down?

40. **Sum or difference (H).** Let a and b be any two irrational numbers. Show that either $a + b$ or $a - b$ must be irrational.

wn Words

41. **Personal perspectives.** Write a short essay describing the most interesting or surprising discovery you made in exploring the material in this section. If any material seemed puzzling or even unbelievable, address that as well. Explain why you chose the topics you did. Finally, comment on the aesthetics of the mathematics and ideas in this section.

42. **With a group of folks.** In a small group, discuss and work through the arguments that the square root of 2 and the square root of 3 are irrational. After your discussion, write a brief narrative describing the arguments in your own words.

43. **Creative writing.** Write an imaginative story (it can be humorous, dramatic, whatever you like) that involves or evokes the ideas of this section.

44. **Power beyond the mathematics.** Provide several real-life issues—ideally, from your own experience—for which some of the strategies of thought presented in this section would provide effective methods for approaching and resolving them.

2.7 Get Real

The Point of Decimals and Pinpointing Numbers on the Real Line

> Why are wise few, fools numerous in the excesse? 'Cause, wanting number, they are numberlesse.
>
> AUGUSTA LOVELACE

▲ Illustration from a 17th-century letter by Felipe Guáman Poma, showing an Incan treasurer holding a quipu. During the 15th and 16th centuries, the Inca used quipus, a system of knotted cords, to record numerical information, such as population and trade with other tribes.

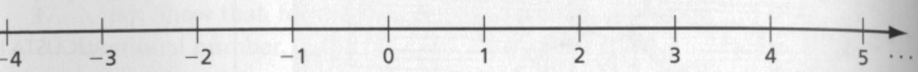
Our development of the notion of "number" took us from the familiar natural numbers and rationals to the more mysterious realm of irrational numbers. While these collections of numbers are distinctive, they all fit together in a basic way: Given any two different numbers, one is bigger than the other. The numbers are all ordered. That orderly hierarchy of numbers by size allows us to represent all numbers on one line. In this final section on number, we explore the connections between the number line and the notion of number.

The guiding principle for this part of the exploration of number is to bring global coherence to separate ideas. By examining the totality of numbers as one entity, we will discover new surprises and develop a better understanding of both the rational and the irrational. Initially some of these discoveries may contradict our intuition. Our exploration involves looking at the types of numbers we know and deducing how those numbers must be interconnected on the number line. This point of view leads to the representation of numbers in decimal form. We must be open-minded and accept logical consequences that we deduce. Once we accept correct conclusions, we will understand the collection of all numbers on the number line—the real numbers—as a coherent idea aptly called the *continuum*.

Lining up

The real number line has appeared in elementary school textbooks as long as school cafeterias have been serving students sloppy joes. Here we start from scratch but soon make unexpected discoveries—just as we did with our sloppy joes—about the familiar idea of the number line.

We begin with the number line itself:



The integer points are labeled, but we would like to be able to label or describe every point on this line. To make progress in this direction, let's consider the points halfway between each consecutive pair of integers. For example, the number 5/2 is the point that sits exactly midway between 2 and 3. In fact, any rational number corresponds to a specific point on this line. For example, we can locate the point to which 37/23 corresponds by dividing each interval between consecutive integers into 23 equal pieces. Then we start at 0 and jump from mark to mark: 1/23, then 2/23, then 3/23, and so on. When we get to 23/23, we see that is the point that is also labeled 1. Jumping 14 more times gets us to 37/23. A similar procedure allows us to find a point on the line corresponding to any rational number.

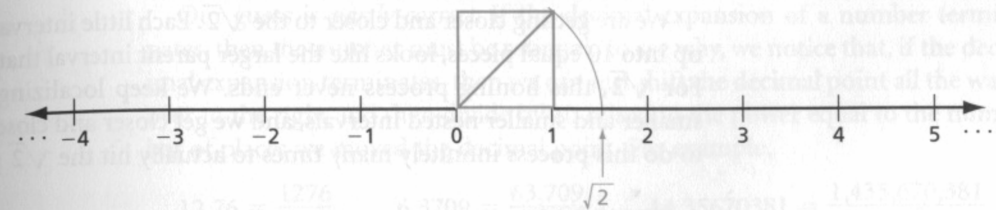
RATIONALS EVERYWHERE

The points associated with rational numbers are all over the line: No matter where we are standing on the line, we can always find a rational number point that is as close as we wish. Suppose we want to find a rational number point that is within a distance of 1/10,000 of where we are standing. We just divide each segment between every two consecutive integers into 10,000 equal pieces and make those 10,000 marks, then mark off all the points that correspond to rational numbers having 10,000 in their denominators (5876/10,000, for example). Therefore, no matter where we are, we will be within 1/10,000 of one of these rational points.

Now that we see that the rationals are essentially everywhere, we may ask: Are there any unlabeled or undescribed points left on our line? The previous section provides us with the answer to this question. Let's construct a point on the number line that definitely does not correspond to a rational number.

AN IRRATIONAL POINT

Here is a way of finding a number on the number line that is not a rational number: Build a square whose base is the interval from 0 to 1. Next draw the diagonal from 0 to the upper-right corner of the square. Using a compass, copy the length of that diagonal onto the number line and make a mark there. What number did we just mark? The square root of 2.



Look for new ways of expressing an idea.

In the previous section we showed that the square root of 2 is irrational. Thus there are points on the line that cannot be labeled with rational numbers, and we are faced with the question: Is there a uniform method to label every single point on the line—rational and irrational?

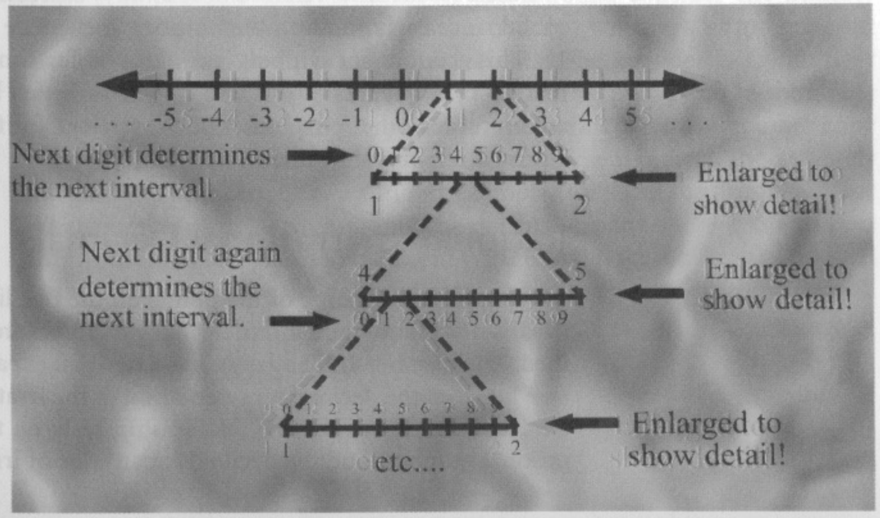
THE DECIMAL POINT

Let's label each point by describing ever more precisely where it sits on our line. This process is familiar, because it is the idea that generates the decimal expansion of numbers. The *decimal expansion* of a number provides us with a road map that allows us to hone in and locate the number on our line. For example, let's consider the decimal expansion of the square root of 2:

$$\sqrt{2} = 1.414213562\ldots.$$

The number to the left of the decimal point tells us that our number will be somewhere between 1 and 2. Where between? We cut the interval from 1 to 2

Stare deeply into the line with your 3D glasses.



PROOF THAT "0.123 . . ." IS IRRATIONAL

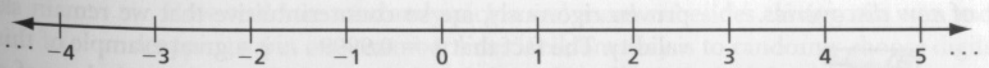
There are only two possibilities: 0.123 . . . is either a rational number or an irrational number. Let us assume it's a rational number. If we could show that this possibility is in fact impossible, that would prove that the other possibility (that the number is irrational) must be true.

If this number were a rational number, its decimal expansion would have to be eventually periodic. In other words, there would have to be a finite string of numbers in the decimal expansion that, from some point on, repeats forever. What would be the length of this finite string that repeats? We don't know. All we are sure of is that the length is finite—let's call that finite length of the repeating pattern L. For example, in the number 45.3219811981198119811 . . . the repeating pattern "1981" has length 4, so here L would be 4.

Recall that our number is created by writing down the natural numbers in order, in juxtaposition. Thus we will eventually write down all numbers that are made just from 1's. In other words, at some point we will see 11 and later 111 and later still 1111 and even later still 11111, etc. Thus we see that there will be long runs of 1's occurring in the decimal expansion. In fact, we will have arbitrarily long runs of 1's. For example, the natural number consisting of a billion 1's in a row will eventually appear. In particular, we can find infinitely many runs of 1's, each of which has more than 10 times L number of 1's in a row (remember that L is the length of the repeating pattern). So at some point the repeating pattern must march through a sea of 1's. The only way that could happen is if the repeating pattern itself were made up exclusively of 1's. If the repeating pattern were all 1's, then from some point onward, all we would see in the decimal expansion of this number would be 1's, contrary to how the number was constructed. This contradiction tells us that our assumption that the number is rational must be false. Therefore, the decimal number 0.1234 . . . is not rational and must be irrational, which completes our proof.

NO HOLES, NO NEIGHBORS

Characterizing the decimal expansions of rational numbers allows us to journey deep into the dense jungle of the real number line. *Gaze closely at the number line. Look really closely, put your nose right up to the page. Stare deeply into the hypnotic line.* Perhaps you are getting sleepy . . . well, snap out of it and wake up! Notice that there are no holes in the number line—instead the line flows smoothly and produces a continuous and unending stream of real numbers—*the continuum.* This "unholey" image leads to a question that has a strange answer.



Suppose we are a particular real number—to ground our thinking, let's suppose we are 0. Now who are our immediate neighbors? In particular, what is the next real number after 0? Can we name it? Suppose someone guessed 1/2. We could easily show that 1/2 is not the next real number after 0; after all, 1/4 is closer to 0 than 1/2. Suppose that someone else guessed 3/702 (= 0.004273 . . .). We could take half of that number and find the number 3/1404 (= 0.002136 . . .), which is even closer to 0. In fact, if anyone gave us any number greater than 0, we could just divide that number in half and find another even smaller number that is also greater than 0. What can we conclude? The answer is that there is no next real number immediately following 0. The moment we specify a number bigger than 0, we could find another number (in fact infinitely many) that is between 0 and the specified number. The real number line flows continuously without breaks and between any two points on the line; we can always find lots of points in between them. Hence, there is no next real number after 0.

Our reasoning could be used to show that there is no next real number after 1 or even after $\sqrt{2}$. We have therefore verified the following:

> **No Next Real Number.**
> *Given any particular real number, there is no next real number immediately following it.*

Redundancy in Representation of Reals

We now have a sense of the connections among points on the number line, their decimal expansions, and the notions of rational and irrational. Every point on the real number line can be represented as a decimal number, but we have not considered the following question:

Is there only one way to write a number in its decimal expansion?

Although it would be convenient for each real number to have only one decimal expansion, unfortunately there are some real numbers that have more than one decimal expansion. We illustrate this fact with an example.

What rational number is 0.999999 . . . ? We'll call this number N (for Nines). Let's answer this question using our method of multiplying by a power of 10, lining up the repeating period, and then subtracting.

$$\begin{array}{r} 10N = 9.9999\ldots \\ -\quad N = 0.9999\ldots \\ \hline 9N = 9.0000\ldots \end{array}$$

So $9N = 9$; and what must N be? $N = 1$. We just proved that $1 = 0.99999\ldots$. Does this equation look strange? Sometimes mathematical results, even when proven rigorously, are so counterintuitive that we remain skeptical of their validity. The fact that $1 = 0.9999\ldots$ is a great example of this phenomenon. Even though we have given a rigorous mathematical proof of this amazing fact, we now give an intuitive argument that may be more convincing.

Suppose we believe that 1 is not equal to 0.99999 . . . (remember that those 9's go on forever without ever stopping). Then one of these numbers would be bigger than the other. Which one would be larger? Certainly 1 would be larger than 0.9999 . . . . If these numbers were not equal, then there must be some numbers between 0.99999 . . . and 1 on the number line. For example, the average of those two numbers would have to be between them. That average must be a number that is larger than 0.99999 . . . and at the same time smaller than 1. What could that number be? It would have to start off with a 0 (otherwise it would not be less than 1). What would the next digit be? It must be a 9 since anything else would make the number smaller than 0.9999 . . . . What about the next digit? It would have to be 9 as well. If we continue in this manner we see that we are building 0.99999 . . . . But that is not bigger than 0.99999 . . . . So there are no numbers between 0.9999 . . . and 1, and hence they must be equal. Did you like this argument? It's amusing to think about.

Here is another way of looking at all those 9's. For some strange reason, people feel comfortable with the fact that 1/3 = 0.33333 . . . and 2/3 = 0.6666 . . . . Well, if we add those together we see 1 = 0.9999 . . . ! We have proved the following:

> **0.9999 . . . = 1.**
> *1 = 0.999999 . . . .*

What is another decimal expansion for the number 0.499999 . . . ? Use the ideas given above and give it a try.

Random Reals

Finally, before closing this section and this chapter we pose an intriguing question:

> If we randomly pick a real number—that is, we take a pin, close our eyes, and place the pin on some point on the real number line—what is the likelihood that the number we picked is a rational number? Is it a 50-50 chance?

A reasonable answer would be 50-50 since a real number is either rational or irrational. Unfortunately, this reasonable sounding answer is far from correct. Although we will not be able to give a rigorous answer to this question until we journey through the world of the infinite in the next chapter, we are able to give a plausible argument that answers the question.

How could we randomly pick a real number besides closing our eyes and dropping a pin on a number line? Well, one way is to randomly choose digits among 0, 1, 2, 3, . . . , 8, 9 and write them down to build a decimal number. We could get a 10-sided die with the sides numbered from 0 to 9. Let's suppose that we always start with 0 so our random number will be between 0 and 1. Now we roll a 10-sided die or have a random number generator spit out digits and we record them:

0.793562845657483883653004836283047261183945737 8 . . . .

We don't stop! We do this forever and thus create a real number. What is the likelihood that this random number is a rational number? Well, for it to be rational, from some point on, the number must have a pattern that repeats forever. But what does that mean? It means that from some point on, we keep repeating the identical pattern without ever deviating. How likely is it that we will repeat a finite pattern forever given that we are generating the digits randomly? The answer is not likely at all—in fact it should "never" happen. There would have to be an amazing and even unheard of conspiracy in the random digits to have them all, from some point on, follow a periodic pattern. Thus the probability that we randomly generate a rational number is actually zero. So if we just randomly pick a real number, it is "certain" to be an irrational number.

Irrationals Abound

This huge preponderance of irrational numbers might be a tough pill to swallow since we are so accustomed and comfortable with rational numbers and since we noticed earlier that the rationals seem to be everywhere on the number line. But, mathematically, rational numbers are actually hard to find. We will see exactly what "probability zero" and "certain" mean in the probability chapter. For now, if we accept the preceding informal analysis, we are faced with an extremely interesting question: If a real number selected at random is "never" a rational number and "always" an irrational number (whatever the notions of "never" and "always" mean), then does that mean that there are, somehow, more irrational numbers than rational numbers? Certainly there are infinitely many rational numbers and infinitely many irrational numbers. Could one of these infinite sets actually be greater than the other? Perhaps what first appears familiar and natural (the rational numbers) will in fact be the exotic and strange, whereas what appeared to be foreign and strange (irrational numbers) will actually turn out to be more the norm! These curious questions set the stage for our next adventure: The world of the infinite.

THE RATIONAL AND IRRATIONAL NUMBERS taken together form the real numbers—the collection of points on a line. We are able to use the decimal expansion of a real number to locate any real number on the number line. The decimal expansion also allows us to distinguish rational numbers from irrational numbers. A number is rational precisely if its decimal expansion eventually repeats; otherwise it is irrational. Using these ideas we are able to devise means of converting repeating decimals to fractions and also to prove that certain real numbers, such as 0.1234567891201112 . . . , are irrational, whereas, surprisingly, 0.9999 . . . = 1. The real line presents a picture of numbers orderly arranged. No number has an immediate neighbor, a number just above it or just below it.

Our strategy for exposing this view of the real numbers was to seek a unified view of all the types of numbers we had developed before. We looked for a relationship that encompassed all the ideas we had generated, in this case, the ideas of rational and irrational numbers. The simple ordering of numbers suggested that we could effectively represent all numbers as points on a line and that we could name each point on the line or number, rational or irrational, using a decimal representation. Some discoveries required us to give up biases and accept logical conclusions. Being open-minded about new ideas is a difficult and important lesson in every arena of life.

. . . an irrational number . . . lies hidden in a kind of cloud of infinity. MICHAEL STIFEL

Seek unifying ideas.

•

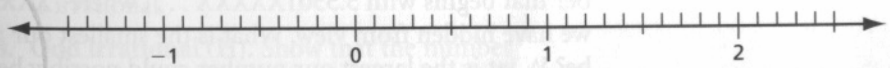
Keep an open mind.

Mindscapes INVITATIONS TO FURTHER THOUGHT

In this section, Mindscapes marked (H) have hints for solutions at the back of the book. Mindscapes marked (S) have solutions.

I. Developing Ideas

1. **X marks the "X-act" spot.**



On the number line above, place an X on the approximate location for each of the following numbers:

$\frac{3}{2}$, $-\frac{1}{3}$, 2.3, −1.1, 0.9, 1.05, −0.55.

2. **Moving the point.** Simplify each of the expressions.

$10 \times (3.14)$, $1000 \times (0.123123\ldots)$, $10 \times (0.4999\ldots)$, $\frac{98.6}{100}$, $\frac{0.333\ldots}{10}$

Beyond Numbers

What Does Infinity Mean?

> The known is finite, the unknown infinite; intellectually we stand on an island in the midst of an illimitable ocean of inexplicability. Our business in every generation is to reclaim a little more land.
>
> THOMAS H. HUXLEY

We seek to put the study of infinity on a firm and logical footing. Since infinity is such a vast and intimidating topic, we prefer to begin by closely looking at the familiar ideas of numbers and counting. We want to count infinite collections, but we don't know how to count that high. Thus we seek different ways to count collections of ordinary objects in the hope that one method might work for infinite collections as well.

Where would we begin to search for the infinite? It's well beyond 1, 2, and 3; certainly past

41359724545124715156120963071048.

Infinity is even beyond

9142452345245001282106162966380902777871210598012670019872665
2198733061118542109827629991176653276533279073847295892835729
8399428310342167161237402372034017487102381084793047109237109
2471098097109230192740192380198231092309182741092730239845235
0293740198230192317240918230471093750193401984019734019824091
7304109740192401984019734091834017236817634651234917864817948.

In fact, as large and incomprehensible as this number is, it is sobering to realize that almost all numbers are far larger still. On the road to the infinite, we would pass this enormous number almost instantly and soon view it as a tiny jot. Even though this number appears early in the infinite list of all numbers, to our minds it seems vast. In reality, we do not even have an intuitive feel for such a large number. The number lacks a sensible name—magnitudes such as millions, billions, or trillions do not make a dent in it. Just reading the digits aloud without error would be a challenge, and finding any of its prime divisors might be nearly impossible.

Even though we are comfortable with large numbers in the abstract, in truth we have little real understanding of such enormous quantities. Given our inability to grasp even these—relatively speaking—modestly sized numbers, is it possible for our minds to fathom the notion of the infinite? Let's first ponder a basic question.

What Does "2" Mean?

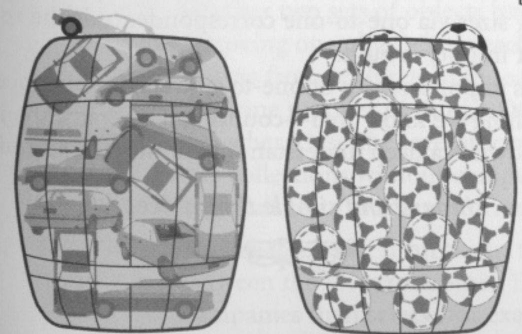
Understand simple things deeply.

Where should we begin our voyage toward the infinite? Our journey to infinity begins with 2. We are all familiar with 2, and we will use that intuitive intimacy to develop a concept of size that will take us well beyond what we know.

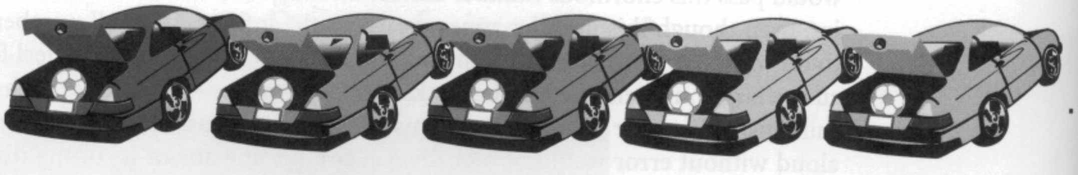
If we have two apples and two hands, we could put one apple in each hand, and no hand would be un-appled, or any apple un-handed. If we have two socks and we put one sock on each hand, then our socks and our hands correspond exactly. This observation also implies that we can put one apple in each sock to demonstrate that the socks and apples also correspond; however, we do not recommend this last experiment unless the socks are clean.

Grappling with infinity requires us to imagine scenarios that are not quite possible but can be clearly conceived in the mind. Suppose we have a huge barrel full of Volvos and a barrel full of soccer balls, and we want to know if there are more Volvos than soccer balls, more soccer balls than Volvos, or the same number of each. How would we decide?

Probably we would just count the number of Volvos and soccer balls in each of the barrels and compare the two numbers. Certainly that method works, but an alternative method allows us to grapple with magnitudes beyond what we can count. Let's devise a method for comparing the barrels without counting Volvos or soccer balls.



We could take one Volvo from the first barrel and one soccer ball from the second, pair them up and put them aside (probably putting the ball in the car's trunk). Then we could pair another Volvo from the first barrel with another soccer ball from the second. If we continue pairing in this fashion, we could tell whether we have the same number of Volvos and soccer balls without ever knowing how many we actually have.



This simple idea is important. We have just described a method for determining when two collections contain the same number of objects without actually counting them. Two collections whose objects can be paired evenly—one from one collection with one from the other collection—are said to have a *one-to-one correspondence.*

Correspondences

As we explore the notion of one-to-one correspondences, we must do something that is extremely difficult: We must forget. We must forget that 8 means something; we must forget that 37 means something. We must strip from our minds the names of numbers, leaving behind only the idea that *two collections of objects are equally numerous, precisely if there is a one-to-one correspondence between the elements of the two collections.* In taking this step, we are moving away from thinking of a number as an attribute of a collection ("I have five fingers on my right hand") to thinking instead of an idea of comparison ("I have just as many fingers on my right hand as on my left, because I can touch all the fingertips on one hand with the fingertips of the other hand").

This focus on comparison, on one-to-one correspondence, allows us to examine the infinite and turn our vague sense of awe into concrete understanding.

a look back

THE FUNDAMENTAL IDEA in the study of infinity is that two collections have the same size if there is a one-to-one correspondence between the members of one collection and the members of the other collection. This compelling concept of comparing sizes via one-to-one correspondence is the rock on which the whole study of infinity is built.

We distilled this important idea of one-to-one correspondence by thinking hard about something we know well—counting. Generally, the most reliable guide to the unknown is a deep understanding of the simple and familiar.

Understand simple things deeply.

3.2 Comparing the Infinite

Pairing Up Collections via a One-to-One Correspondence

▲ *The Creation of Adam,* Sistine Chapel (1508–12) by Michelangelo. An early one-to-one correspondence

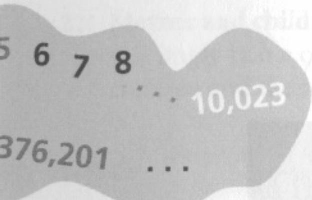
> I saw . . . a quantity passing through infinity and changing its sign from plus to minus. I saw exactly how it happened . . . but it was after dinner and I let it go.
>
> SIR WINSTON CHURCHILL

We now enter the world of infinity armed with one idea—a criterion for comparison: the one-to-one correspondence. We will test the consequences of this idea by comparing some infinite collections of familiar objects. As usual, our most productive strategy is to examine the familiar before we journey toward the unknown.

Since numbers are really the only infinite collections we know, we turn to them for our first examples to help us become accustomed to the idea of one-to-one correspondence. We start with the most basic collection of numbers we can think of and then consider related but different collections. Our goal is to determine whether various collections can be put into one-to-one correspondence, since one-to-one correspondence is the fundamental principle on which our investigation of infinity is built.

Familiar but Infinite

What is familiar and concrete to one person may be foreign and abstract to another, but as far as numbers are concerned, we probably all agree on which are the most familiar. In 1886, Leopold Kronecker, a number theorist, made a statement about what is basic in the world of mathematics: "God created the positive integers; all the rest is human creation."



One, two, three, . . . these are the positive integers. For every positive integer, there is a next bigger one. Although we may think of these positive integers successively, we may also think of all of them at once—that is, think about the collection of all positive integers. The collection (also referred to as the *set*) of positive integers is so basic and natural to our way of thinking that it is called the set of *natural numbers*.

The set of all natural numbers is our first infinite set, and it has a comfortable feel about it. Among infinite sets, the natural numbers seem the most natural. By examining this and related collections of numbers, we will begin to develop a better and more precise idea of infinity.

NATURAL NUMBERS WITH 1 REMOVED

Suppose we are given another copy of the set of natural numbers—in a different font:

1, 2, 3, ...

Unfortunately, we absentmindedly lost the number *1*. Thus, this new set is the collection of natural numbers with the number *1* removed. Specifically, our set consists of

2, 3, 4, ...

Now we have a new infinite set. But it has one less element than the set of natural numbers . . . or has it?

On the one hand, we can observe through life experience that, if we have a barrel of Volvos and one is removed, we then have fewer Volvos left. It seems reasonable to conclude that we have fewer natural numbers if we remove the number *1*. On the other hand, we may think, "Hey, infinity is infinity is infinity, so the new set doesn't contain fewer elements."

Our intuition is pulled in two directions. One direction is the "infinity is infinity" camp; the other is the "take one away, you have one less" school. We will soon discover that both these arguments will lead us astray. What's wrong with our intuition? Nothing, except that our insights and life experiences involving collections of everyday objects will not always apply to infinite collections.

If our intuition leads to two opposite conclusions and both sound reasonable, we are compelled to investigate until we understand the consequences of both ideas. If we believe that the set of natural numbers with the number 1 removed is the same size as the set of natural numbers, then we need a rigorous and logical reason. Vague thoughts of "infinity is infinity" will not suffice in a quantitative court of law. However, we must remember to avoid little distractions, such as our entire life history, that tell us when we remove an object, we are left with fewer things. Let's keep an open mind and remember that our criterion for determining the equivalence of two collections is not a vague, undefined feeling developed through years of experience but, instead, is a clear criterion stated crisply and explicitly as the existence of a one-to-one correspondence. Since we have formulated an explicit definition, let's rely on it in preference to general impressions. Infinity is a large, wild beast; but, if we remain focused on our principle of comparison, we will have infinity tamely eating out of the palm of our hand.

A Search for a One-to-One Correspondence

To determine whether there are as many natural numbers as there are integers starting with *2, 3, 4, . . .* , we must ask if there is a one-to-one correspondence between the elements of the set

2, 3, 4, . . .

and those of the set of natural numbers

1, 2, 3, ...

If we are from the school of "take one away, we have one less," it may appear that there cannot be a one-to-one correspondence between our two sets, since, if we paired the numbers in the two sets, we'd see the following:

Natural Numbers	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19 . . .
A Pairing		↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕
Our New Set		*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19 . . .*

Just because a specific attempt failed does not mean that the task at hand is impossible.

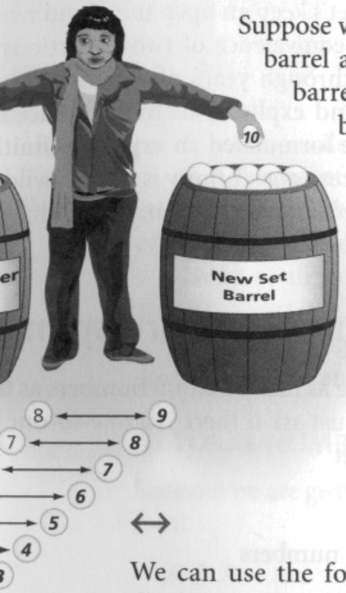
Notice how the "1" in the natural numbers is alone and is not paired with any number in our new set. Hence this pairing is not a one-to-one correspondence. Does this failure imply that no one-to-one correspondence exists? The answer is a resounding no! A one-to-one correspondence may still exist. We merely conclude that the *particular* pairing we just created isn't a one-to-one correspondence.

THE NATURALS MINUS 1 EQUALS THE NATURALS

It turns out that, in fact, a one-to-one correspondence between these two sets does exist, which will not surprise members of the "infinity is infinity" camp.

We illustrate such a one-to-one correspondence below. But members of the "infinity is infinity" camp should not be too smug just yet.

al Numbers	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19...
ew Pairing	↘	↘	↘	↘	↘	↘	↘	↘	↘	↘	↘	↘	↘	↘	↘	↘	↘	↘	↘
ur New Set		*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19* *20*...



Suppose we dump all the natural numbers into the Natural Number barrel and the natural numbers with *1* removed in the New Set barrel. In the new pairing, we grab 1 from the Natural Number barrel and **2** from the New Set barrel, pair them, and toss them. Next, we pair 2 with **3**, then 3 with **4**, 4 with **5**, and so on.

Notice that every element from the Natural Number barrel is paired with exactly one of the elements from our New Set barrel, and each number from our New Set barrel is associated with exactly one natural number. The moment we mention a particular number from one list, we know who it pairs with from the other list. After completing this infinite process, no numbers remain; both barrels are empty.

$\leftrightarrow$

We can use the following symbol to express this one-to-one correspondence: $\leftrightarrow$. The symbol compactly illustrates which numbers are paired. Thus, $n \leftrightarrow n + 1$ means that the number n from the Natural Number barrel is paired with the number $n + 1$ from the New Set barrel. Once we know which number n represents, we immediately know its mate is $n + 1$.

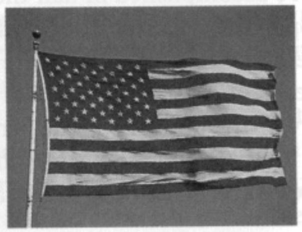
For example, if n is 4, then we see its mate is ***4 + 1***, which is **5**. So 4 from the Natural Number barrel is paired with **5** from the New Set barrel. Notice that this pairing is exactly the one we described in the picture.

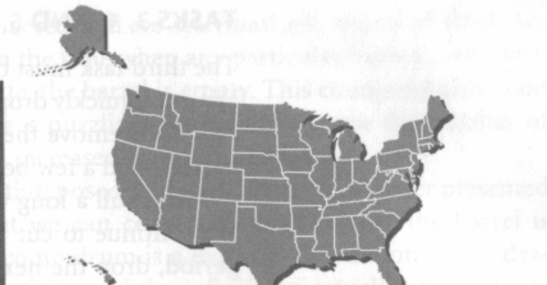
CARDINALITY

The existence of this one-to-one correspondence means that these two sets have the same number of elements. We must be careful here. We really should not say that these sets have the same "number" of elements, since infinity is not actually a number. How can we get around this thorn? We create new terminology. We use the phrase *cardinality of a set* to mean the "number" of things in the set, with the understanding that the set may contain infinitely many things. If a set contains only finitely many things, then its cardinality is just the number of things in the set.

Two sets have the *same cardinality* if there is a one-to-one correspondence between the elements of one set and the elements of the other set—*such as* the set of stars on the U.S. flag and the set of states in the United States. *In this* case, both sets have finite cardinality, and that cardinality equals 50.

How would you show a one-to-one correspondence between the set of stars and the set of states?

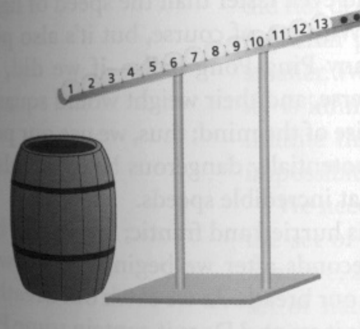




As an example involving infinite collections, we saw that the set of natural numbers and the set of natural numbers greater than 1 have the same cardinality.

The Ping-Pong Ball Conundrum

Let's now consider a story that frees us from the confines of our physical reality and allows us to use our imagination. At the end of the story, we pose several questions, and we encourage you to take a moment to guess an answer.



The scene opens with a very large barrel center stage on a raised, angled ledge. Immediately to the right of the barrel, lined up like little round soldiers, we see an unending row of white, numbered Ping-Pong balls ordered sequentially starting with 1. There is one Ping-Pong ball for each natural number.

Now we embark on a mental experiment. This experiment will last for exactly one minute, and in that time we will perform an infinite number of tasks. But 60 seconds after we start, our stopwatch's alarm will beep, and we will stop—period.

TASK 1

We begin with 60 seconds on the clock. In half that time, 30 seconds, we must pour the first 10 Ping-Pong balls (numbered 1 through 10) into the large barrel and then reach into the barrel, find the Ping-Pong ball numbered 1, and remove it (just throw it away). We are left with nine balls in the barrel and one thrown away.

TASK 2

Now we have 30 seconds remaining, and we begin to pick up the pace. In half the time remaining, 15 seconds, we must dump into the barrel the next 10 Ping-Pong balls (numbered 11 through 20) and then reach in and scoop out the ball numbered 2. We are now left with 18 balls in the barrel and two thrown away. But we don't stop yet.

TASKS 3, 4, AND 5

The third task must be accomplished in half the time remaining, 7.5 seconds. We must quickly drop in the next 10 balls (numbered 21 through 30) and then find and remove the ball numbered 3. We are then left with 27 balls in the barrel—and a few beads of perspiration on our foreheads.

We're still a long way from finished, but perhaps the pattern is emerging. We continue to cut the time remaining in half and, in each individual half period, drop the next 10 balls into the barrel and fish out the ball with the lowest number.

TASK 6

At the sixth stage, we drop the balls numbered 51 through 60 into the barrel and remove the ball numbered 6, leaving 54 balls in the barrel. We have to work fast, because this entire stage must be completed in 0.9375 of a second (that is, half of half of half of half of half of 60 seconds).

REMAINING TASKS

Clearly, we must pick up the pace. In fact, we see that soon we will have to move faster than the speed of sound and even faster than the speed of light—now that's fast. Physically impossible? Why yes, of course, but it's also physically impossible to have infinitely many Ping-Pong balls—if we did, they would take up all the space in our universe, and their weight would squash us like small bugs. Happily, this is an exercise of the mind; thus, we use our power of imagination to save us from these potentially dangerous balls and also to allow us to dump in and pull out balls at incredible speeds.

Our experiment of the imagination is hurried and frantic; mercifully, however, it lasts only one minute. Sixty seconds after we begin, the stopwatch beeps and we stop and attempt to catch our breath. As we catch our breath, we glance into the barrel. What do we see? Is it empty? Does it contain some Ping-Pong balls? Does it contain infinitely many Ping-Pong balls? We invite you to consider these questions.

ON THE BALL

Do Ping-Pong balls remain in the barrel after the minute has expired? If so, then we have an annoying request: Name one! Any ball remaining has a number stamped on its surface. What is that number? Recall that the balls are numbered 1, 2, 3, . . . forever—there is no last ball. What is the number of a ball in the barrel? Could it be 5? No, because we know exactly when the ball numbered 5 was removed—namely, at the fifth stage (remember, we dumped balls 41–50 into the barrel and removed the ball numbered 5). Could ball 45,671,803 remain? Well no, because we know exactly when we removed it from the barrel—specifically, at the 45,671,803rd stage of the experiment. Thus, which balls are left? The answer is amazing and surprising: "NONE"!

Since all the balls were numbered and we systematically removed them, we can state with exact precision the time when any particular ball was removed. Thus, at the end of the minute, the barrel is empty. This counterintuitive and perhaps unbelievable answer is puzzling—especially because the number of Ping-Pong balls in the barrel increased by nine at each stage.

This experiment, the question posed, and particularly the answer presented require serious thought, but we can convince ourselves that the barrel is empty. This Ping-Pong ball conundrum is a dynamic illustration of the dramatic difference between the finite and the infinite. In actuality, this activity produced a one-to-one pairing between the intervals of half times (stages) and the numbered balls. We were able to pair them evenly. Again, our intuition, based on finite collections, is not always accurate when applied to infinity. Nevertheless, through reason, the experience we are gaining will help us develop an understanding of infinity. To further develop this understanding, let's explore other infinite collections.

Looking for Giants

When we saw that removing the number *1* from the set of natural numbers did not decrease the size or the cardinality of the set, we surprised those whose intuition previously dictated that removing an element should make a set smaller. We will also decimate the intuition of those who remain in the "infinity is infinity" camp by demonstrating that some infinite sets are even more infinite than the natural numbers! How can we even begin this seemingly impossible quest for giant sets?

We need to think of some infinite sets that are likely to be even larger than the set of natural numbers. A possibility is the set of *all* integers: positive, negative, and zero. This set contains infinitely many more numbers than the set of natural numbers—namely, all the negative integers and zero—so it appears to be a good candidate. Intuitively, there appear to be *twice* as many integers as natural numbers.

INTEGERS EQUAL NATURALS

Unfortunately, merely adding infinitely many negative numbers and zero is not enough to increase the size of the set of natural numbers. To prove this statement, as always, we need to exhibit an explicit one-to-one correspondence between the set of natural numbers and the set of all integers. You can find that correspondence yourself. To describe such a correspondence, draw a table that lists the natural numbers, 1, 2, 3, . . . , down the left side of the paper. Next to each natural number, write one integer (positive, negative, or zero) in such a systematic fashion that all integers will eventually appear on the list. Wait to look at our answer until you've given it a try.

Natural Numbers		All integers
1	⟷	0
2	⟷	1
3	⟷	−1
4	⟷	2
5	⟷	−2
6	⟷	3
7	⟷	−3
…	…	…
$2n$ (The even numbers)	⟷	n
$2n + 1$ (The odd numbers)	⟷	$-n$
…	…	…

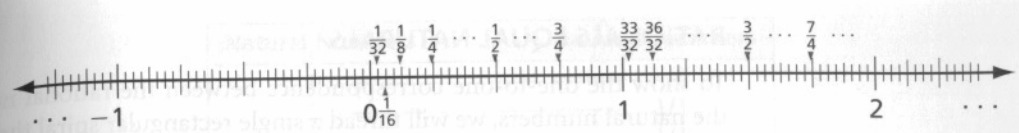
Notice that in this pairing, every natural number appears exactly once in the left-hand column, and every integer (positive, negative, or zero) appears exactly once at some time in the right-hand column. Thus, we have an even pairing, a one-to-one correspondence. What number is paired with −9 in the right column? What number is paired with 9 in the left column? We can choose random elements from either set and figure out under this pairing with whom those random elements get paired—this experimentation is the best way to get a feel for the correspondence. Is any number not paired? No.

So, we have proven that the set of natural numbers has the same cardinality as the set of all integers. Notice how the two symbolic expressions toward the bottom of our list capture the spirit of the pairing; namely, the even natural numbers are paired with the positive integers, whereas, the odd natural numbers are paired with the negative integers or zero. For example, the natural number 32 can be written as 2×16, so the symbolic expression $2n \leftrightarrow n$ shows us that 32 is paired with 16. However, 33 can be written as $(2 \times 16) + 1$; thus, the symbolic expression $2n + 1 \leftrightarrow -n$ shows us that 33 is paired with −16. So, if we looked down our list, at some point we would see the following pairings:

32	⟷	16
33	⟷	−16

Rational Numbers

We might feel more or less stymied in our quest for an infinite set bigger than the set of natural numbers. But let's not forget that lots of numbers are not integers. For example, how about the rational numbers? Recall that the set of rational numbers is the set of all ratios of integers (fractions). Between any two integers there are infinitely many rationals. Within the set of rational numbers, we actually have infinitely many different infinite sets. The set of rational numbers must be huge.



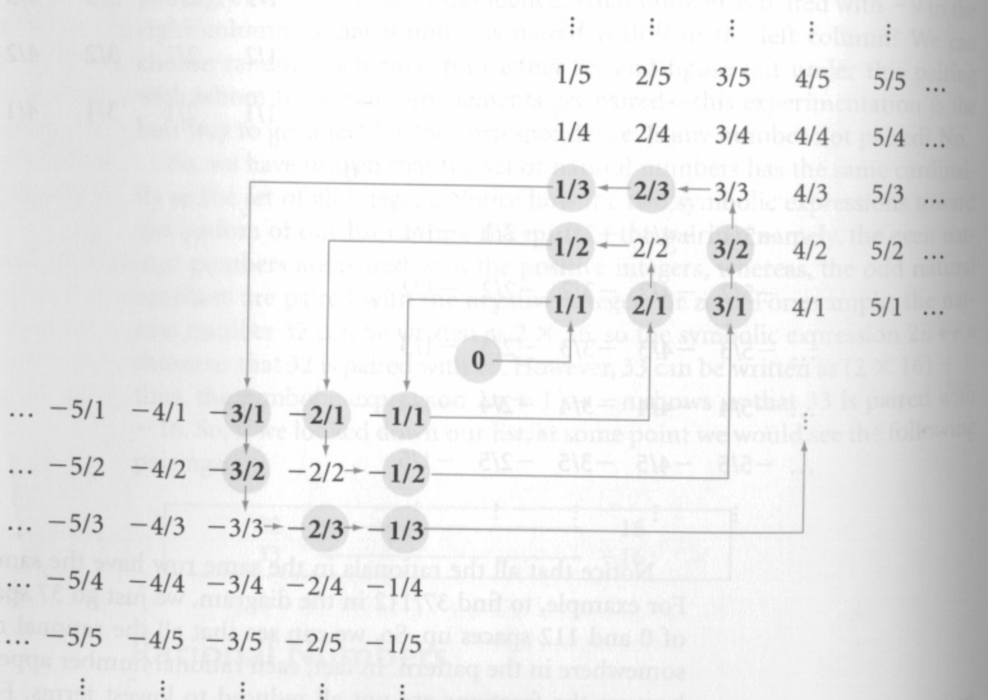
Unfortunately, in our search for different sizes of infinity, the rationals are still not numerous enough. We rely once again on the definition of same cardinality; namely, two sets have the same cardinality if there is a one-to-one correspondence between the elements of one set and the elements of the other. The trick here is to write down the rational numbers in a convenient and systematic way so that we know we have listed them all. The idea is to put all the rational numbers with numerator 1 in one column, all those with numerator 2 in another column, and so on, as shown in the following diagram.

						⋮	⋮	⋮	⋮	⋮
						1/5	2/5	3/5	4/5	5/5 …
						1/4	2/4	3/4	4/4	5/4 …
						1/3	2/3	3/3	4/3	5/3 …
						1/2	2/2	3/2	4/2	5/2 …
						1/1	2/1	3/1	4/1	5/1 …
					0					
… −5/1	−4/1	−3/1	−2/1	−1/1						
… −5/2	−4/2	−3/2	−2/2	−1/2						
… −5/3	−4/3	−3/3	−2/3	−1/3						
… −5/4	−4/4	−3/4	−2/4	−1/4						
… −5/5	−4/5	−3/5	−2/5	−1/5						
⋮	⋮	⋮	⋮	⋮						

Notice that all the rationals in the same row have the same denominator. For example, to find 37/112 in the diagram, we just go 37 spaces to the right of 0 and 112 spaces up. So, we can see that all the rational numbers appear somewhere in the pattern. In fact, each rational number appears many times, because the fractions are not all reduced to lowest terms. For example, 1/2 appears and so does 2/4, 3/6, and so on, but that redundancy is okay, because, at this point, we merely want a systematic method of recording every rational number without omitting any. Notice that all the positive rational numbers appear in the upper-right part of the diagram, all the negative rationals appear in the lower left, and 0 is in the middle. So far, we have described a way of writing down all the rational numbers.

RATIONALS EQUAL NATURALS

To show the one-to-one correspondence between the rational numbers and the natural numbers, we will thread a single rectangular spiral through all the rationals, starting in the middle at 0 and moving counterclockwise outward. To see the one-to-one correspondence with the natural numbers, we will count the rational numbers as we encounter them along the spiral and make them bold-face to remind us that we have paired that rational with a natural number. We start with the rational **0** corresponding to the natural number 1; then, moving to the right and up, the rational **1/1** = 1 corresponds to the natural number 2; the rational **−1/1** = −1 corresponds to 3; the rational **2/1** = 2 corresponds to 4. We next come to 2/2, which has already been counted, so we skip it and move to **1/2,** which corresponds to 5; then **−2/1** = −2 corresponds to 6. We skip −2/2, since it equals −1, which already corresponds to 3, and move to **−1/2,** which corresponds to 7; and so on. Notice that every rational number will eventually be reached and put in correspondence with some natural number. This one-to-one correspondence shows that the set of all rational numbers has the same cardinality as the set of the natural numbers.



We may now straighten out the spiral and produce the following list showing the one-to-one correspondence.

Notice that if someone examined just the list, he or she would have difficulty detecting the pattern, which arises from the spiral previously illustrated.

Natural Numbers	Rational Numbers
1	0
2	1/1
3	−1/1
4	2/1
5	1/2
6	−2/1
7	−1/2
8	3/1
9	3/2
10	2/3
11	1/3
12	−3/1
…	…

Threading and counting along the spiral provides an important insight into sets containing the same cardinality as the set of natural numbers. If we can write a set as an infinite list, we can make a one-to-one correspondence with the natural numbers.

We now see that the rational numbers did not provide us with an infinity larger than that of the natural numbers. Our quest for an even grander infinity has thus far failed. But perhaps in chasing sets larger than an infinite set, we should expect to have to go a long, long way.

a look back

Two sets have the same cardinality if there is a one-to-one correspondence between the contents of one and the contents of the other.

The set of natural numbers, 1, 2, 3, . . . , is a natural first infinite set to investigate. The set of natural numbers has the same cardinality as the set of natural numbers with the number *1* removed; the same cardinality as the set of all integers; and even the same cardinality as the set of all rationals. One might naturally, but mistakenly, guess that all infinite sets have the same cardinality as the natural numbers.

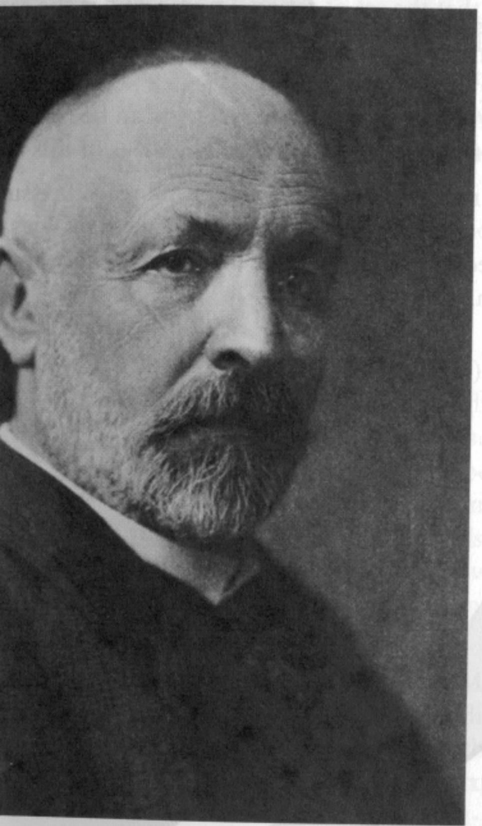
Our strategy for understanding a difficult topic is to explore and experiment with the simplest, most familiar example we can find. In this case, we compared the natural numbers with variations of the natural numbers (the natural numbers minus 1, the integers, the rationals). As we make discoveries that are counterintuitive, we gain experience that will retrain our intuition. The discovery of counterintuitive truths liberates us to think of notions grander still.

Experiments with the familiar help us to understand the unknown.

e first
nity.

The Missing Member

Georg Cantor Answers: Are Some Infinities Larger Than Others?



From the paradise created for us by Cantor, no one will drive us out.

DAVID HILBERT

Around 1872, the German mathematician Georg Cantor shook the foundations of infinity when he proved that the set of real numbers has more elements than the set of natural numbers. In other words, he proved that infinity is not one size but that some infinities are more infinite than others. At first such a notion might seem nonsensical. Once we have reached infinity, surely we cannot climb farther. But Cantor showed that there were yet higher mountains to scale.

To show that the real numbers are more numerous than the natural numbers, Cantor focused intently on what it would mean for the real numbers and the natural numbers to have the same cardinality. It would mean that the real and natural numbers could be put in one-to-one correspondence. Writing down what such a correspondence might look like gives us a visual clue as to why any attempted correspondence between the natural numbers and the real numbers could not include every real; some real is missing—*the missing member.*

From Bizarre to Intuitive

Consider the counterintuitive.

When Cantor conceived his idea near the end of the 19th century, many mathematicians resisted it strongly and attacked Cantor in a personal, abusive manner. These attacks, combined with other psychological issues, contributed to a bleak existence for Cantor, who spent much of the last part of his life in an insane asylum. People do not easily give up their intuition or beliefs, and the resistance of mathematicians to Cantor's "taming" of infinity was neither the first nor the last time that people resisted ideas that contradicted their preconceptions.

In 1637, Galileo was imprisoned for saying that the Earth moves around the Sun. The idea that the Earth moves is extremely counterintuitive. We accept the idea of a moving Earth principally because everyone else believes it. When we were children, we were told that the Earth moves, and so a moving Earth does not present a challenge to our beliefs. In fact, anyone who would today assert that the Earth is stationary and the Sun revolves around the Earth would be regarded as a kook—and for good reason. However, historically speaking, a moving Earth was not easy to prove. Galileo was imprisoned because authorities of the time were not able to see how they could preserve their belief in the centrality of humanity and accept the radical idea that the Earth was just one of several planets revolving about the Sun.

Most people do not consider infinity to be of life-threatening importance. Cantor was not imprisoned for heretical thinking. Infinity is a little too "out there" for most people to get worked up about. But, for those mathematicians who were deeply immersed in such issues, the idea of having many different infinities was a tremendous blow. Out of vague and ill-defined notions of sizes of sets, Cantor distilled the fundamental idea of one-to-one correspondence. This idea is so basic that one feels compelled to accept its logical consequences. But these consequences contradict the intuition of most people—until their intuition is adjusted, after which the concept of more than one infinity becomes perfectly natural. Thanks to Cantor, who reached out and considered the counterintuitive, no mathematician today has a problem encompassing the idea of multiple infinities.

Sometimes understanding a fact requires us to change our minds in a dramatic way. However, that initially counterintuitive fact may at some future date attain the level of intuitive truth. For the Greeks, the existence of irrational numbers presented such a challenge. For a century now, mathematicians have come to understand the hierarchy of infinities. Time and again the bizarre and rare, after their discovery and assimilation, become the natural and familiar. These mental transitions are some of the great joys of thought.

Unequal Decimals

We now turn to the task of demonstrating that the set of real numbers has a strictly larger cardinality than the set of natural numbers. In other words, we now show that the set of real numbers is more infinite than the set of natural numbers.

Recall that each real number can be expressed as an infinitely long decimal expansion. For example,

243.476666875446800887672875849345788445321 . . .

is a real number. Before moving forward, we must first make an easy observation about real numbers. Suppose we examine two decimal numbers, but we cover all the digits with question marks, except for the digit in the fifth place after the decimal point. So, we have two funny looking numbers: ??.????2???? . . . and ??.????4???? . . . . We can't identify these numbers, because we can read only the fifth digit after the decimal point. But one thing we do know is that these two numbers are different. If they were the same, we could not have a 2 in the fifth place after the decimal point of one number and a 4 in the fifth place in the other. Likewise, if we have two numbers and one has a 2 and the other has a 4 in the 87th place after the decimal, then the two numbers must be different. This observation is not hard to understand, but it is a key to Cantor's reasoning.

TWO LONG LISTS

Cantor proved that there are more real numbers than natural numbers through a clever, yet simple, idea. His basic strategy was to attempt an impossible task in order to understand why it couldn't be done. If the set of real numbers and the set of natural numbers have the same cardinality, then there would be a one-to-one correspondence between them. So his idea was to list the natural numbers down the left-hand side of a page, list reals in the right-hand column, and then show how to construct a real number that could not appear on the list. He showed that, once we commit ourselves to a list of reals in the right-hand column, one real number corresponding to each natural number, then we can describe a real decimal number that does not appear anywhere on that infinite list. So, we could not have listed all the real numbers in the right-hand column. Thus, the natural numbers and the real numbers could not be put in one-to-one correspondence, and so there are more real numbers than natural numbers.

Imagine a barrel containing all the natural numbers and another barrel containing all the real numbers. We will now reach in and grab the natural numbers one at a time and in order, pairing each with a real number we grabbed from the other barrel. We will then record the pairing, grab another two, and repeat. This procedure creates a list of all the natural numbers in one column and a list of real numbers in the other column. To illustrate this process, the pairing might begin like this:

Natural Numbers		Real numbers
1	⟷	0.55627363495617384921348 . . .
2	⟷	142.02732981638472734718734 . . .
3	⟷	7.61235987364823519197234 . . .
4	⟷	238.18521936478912092519027 . . .
5	⟷	−0.00083738265191836548713 . . .
6	⟷	31.84722235675444566903346 . . .
7	⟷	658.3333333333543356708632 . . .
8	⟷	−37.83958382139857446882891 . . .
⋮		⋮
11	⟷	29.99907982742111199853769 . . .
⋮	⋮	⋮

We can view this correspondence as two infinitely long columns: On the left we have a complete list of all natural numbers, and on the right we have a list of real numbers. We are now wondering whether every single real number will appear somewhere in the right-hand list. If the set of real numbers and the set of natural numbers have the same cardinality, then it would be possible to list all the reals in some order—one for each natural number. But, in fact, we will construct a real number in decimal form that does not appear anywhere in the right-hand column. That is, we will show that there are so many more real numbers than natural numbers that given *any* pairing between the natural numbers and the reals, a real number will always be left out—it is impossible to produce a one-to-one correspondence. Put another way, if we have the natural numbers in one barrel and the real numbers in another barrel, and we remove one natural number and one real number, pair them, and repeat, then after we run out of natural numbers, there will be real numbers left over! In fact, most of the real numbers would still be left in the barrel.

A MISSING REAL

We are going to write down a particular real number that we will call M, for "missing." We will write it in its decimal expansion. Our number M will be between 0 and 1, so its decimal expansion begins with 0.??? . . . . Now we must decide what the digits "??? . . ." are. Each digit will be one of two possibilities:

M = 0.??? . . .
? = 2 or 4

A general description of M.

If 1st Real = ?.2?? . . . ,	If 1st Real = { ?.0?? . . . , ?.1?? . . . , ?.3?? . . . , ?.4?? . . . , ?.5?? . . . , ?.6?? . . . , ?.7?? . . . , ?.8?? . . . , ?.9?? . . . }
then M = 0.4?? . . .	then M = 0.2?? . . .

Determining the first digit of M by considering the first digit of the first Real number on our list. Either way we know that M and the real number are not equal because the first digit after the decimal point is different.

a 2 or a 4. We will decide on the digits of our number *M* one at a time, successively, so we must be patient. We now describe the criterion by which we choose each digit of our number *M*.

We start with the first digit after the decimal point. Remember that we have a table that pairs one real number with each natural number. So some real number is paired with the natural number 1. We look at the first digit following the decimal point. Although this insight will not shake the very foundations of your universe, we boldly state that there are only two possibilities for that first digit: It is either 2 or it is not 2. We will use the first digit of that first real number to decide on the first digit of our number *M*—the real number we are building. If the first digit of that first real number is 2, then we will set the first digit after the decimal point of our number *M* to be 4. If, however, the first digit of that first real number is not 2, then we will set the first digit after the decimal point of our number *M* to be 2. Observe that, no matter what digits come next in *M*, we know for sure that the number *M* will not equal the real number paired with 1. Why? Because *M* and the real number paired with 1 have different first digits after the decimal point!

How will we define the digit that is in the second place after the decimal point of our number *M*? We take a look at the real number paired with the natural number 2, see what its second-place digit after the decimal point is, and ask if it equals 2. If that digit is 2, then we will set the second digit of our number *M* to be 4. If, however, that digit is not 2, we will set the second digit of our number *M* to be 2. Notice that we have defined *M* such that *M*'s second digit after the decimal point is not the same as the second digit after the decimal point of the real number corresponding to 2. In particular, *M* cannot equal the second real number in the list—the real number corresponding to 2.

We continue to define the digits of *M* in this fashion. So, for example, to determine the 11th digit of *M*, we look at the 11th digit in the real number that is paired with the natural number 11. If that digit is 2, then we define the 11th digit of our number *M* to be 4; if that digit is not 2, then we define the 11th digit of our number *M* to be 2.

SO FAR, SO GOOD?

To determine whether this process is clear, let's look at the lists of natural numbers and real numbers in the previous table and write down what *M* would be. The answer appears in the next paragraph, so write your answer first. Why is Cantor's argument referred to as Cantor's *diagonalization proof?*

If the one-to-one correspondence is the one given in the preceding section, then *M* would equal 0.24442424 . . . . Incidentally, *M*'s 11th digit after the decimal point is 4. Notice that, no matter what the given correspondence is, the number *M* will have only 2's and 4's in its decimal expansion.

IS M REALLY MISSING?

At this point, we know how to construct *M* if we are given a table with natural numbers in one column, each corresponding to a real number. Of course, *M* is a real number. Could *M* appear anywhere on this list of reals? If so, where could it appear?

Is *M* the first real number on the list—the real number that corresponds to 1? No, because we selected the first digit of *M* so that it differs from the first digit of the real number paired with the natural number 1.

Is *M* the second number on the list? No, because we selected the second digit of *M* so that it differs from the second digit of the number associated with the natural number 2.

Is *M* the 1,582,987th number in the list? No, because we selected the 1,582,987th digit of *M* to differ from the 1,582,987th digit of the real number paired with the natural number 1,582,987.

What does all this mean? *M* cannot equal any real number in the table; therefore, *M* is not on the list! If we had entered different reals, then we would build a different *M*. But the point is that, for any particular attempt we make to list the natural numbers on the left and correspond a real number with each natural number, we can create a decimal number *M* that is not on that list. In other words, it is impossible for any correspondence of natural numbers and reals to contain all the real numbers. In particular, there is no one-to-one correspondence between the natural numbers and the reals!

~~Infinity~~► Infinities

We have just shown that, if we are given any correspondence from the natural numbers to the real numbers, then we can produce a real number that does not correspond to any one of the natural numbers. So, no matter how we try to pair numbers from the two sets, we will always have real numbers left over. Therefore, there are **different sizes of infinity!** We have found two different infinities: The cardinality of the natural numbers is **not the same** as the cardinality of the real numbers, even though both sets are infinite. There are more real numbers than there are natural numbers, or, phrased another way, the cardinality of the real numbers is larger than the cardinality of the natural numbers, even though the natural numbers are already infinite. Incredible!

> **Cantor's Theorem.**
> *There are more real numbers than natural numbers.*

The idea of infinities being larger than other infinities is not easy to digest. We must think about it, work through the preceding argument many times over, and explain it to someone. It is a great challenge to try to understand something that appears to contradict our personal hunches and intuition, especially for members of the now defunct "infinity is infinity" camp. We must master the logical arguments so solidly that they become irrefutable. Only then will our preexisting biases give way to a new idea of the infinite. This process requires much effort, but, is well worth the investment.

a look back

CANTOR'S DIAGONALIZATION ARGUMENT shows that, for any given correspondence from the natural numbers to the reals, we can always construct a new real number that does not appear on the list—that is, that does not correspond to any one of the natural numbers. Consequently, the cardinality of the real numbers is not the same as the cardinality of the natural numbers. There are more real numbers than natural numbers.

What strategy allowed us to discover a proof that there are more real numbers than natural numbers? We began by looking carefully at what must be true if the real numbers and the natural numbers had the same cardinality. We wrote down a possible correspondence—naturals on the left, reals on the right—and thought about the question: Are any real numbers missing? We noticed that real numbers have infinitely many digits past the decimal point, and for two real numbers to be different, they need to differ only in one of those places. Keeping this in mind while looking at the lists led us to the diagonalization argument.

When we feel that something is impossible, or if we just don't know whether it is true or false, a good way to find the truth is to explore carefully what would happen if the impossible or unknown were true. Systematically design what-if scenarios and play them to their conclusions.

I can see it, but don't believe it! GEORG CANTOR

Explore consequences of possibilities.

Mindscapes INVITATIONS TO FURTHER THOUGHT

In this section, Mindscapes marked (H) have hints for solutions at the back of the book. Mindscapes marked (S) have solutions.

I. Developing Ideas

1. **Shake 'em up.** What did Georg Cantor do that "shook the foundations of infinity"?

2. **Detecting digits.** Here's a list of three numbers between 0 and 1:

   0.12345
   0.24242
   0.98765

   What's the first digit of the first number? What's the second digit of the second number? What's the third digit of the third number?

3. **Delving into digits.** Consider the real number

   0.12345678910111213141516 . . . .

   Describe in words how this number is constructed. What's its 14th What's the 25th digit? What's the 31st digit?

4. **Undercover friend.** Your friend gives you a list of three, five-digit but she only reveals one digit in each:

   3????
   ?8???
   ??2??

   Can you describe a five-digit number you know for certain will no her list? If so, give one; if not, explain why not.

5. **Underhanded friend.** Now your friend shows you a new list of th digit numbers, again with only a few digits revealed:

   6????
   ?5???
   ?????

   Can you describe a five-digit number you know for certain will no her list? If so, give one; if not, explain why not.

II. Solidifying Ideas

6. **Dodge Ball.** Revisit the game of Dodge Ball from Chapter 1: "Fun Games." Play it several times with several people. Get the strategy then explain to your opponents the underlying principle. Record t of the games.

7. **Don't dodge the connection (S).** Explain the connection between Ball game and Cantor's proof that the cardinality of the reals is gr the cardinality of the natural numbers.

8. **Cantor with 3's and 7's.** Rework Cantor's proof from the beginnir time, however, if the digit under consideration is 3, then make the sponding digit of M a 7; and if the digit is not 3, make the associat M a 3.

9. **Cantor with 4's and 8's.** Rework Cantor's proof from the beginnir time, however, if the digit under consideration is 4, then make the sponding digit of M an 8; and if the digit is not 4, make the associ of M a 4.

10. **Think positive.** Prove that the cardinality of the positive real num same as the cardinality of the negative real numbers. (*Caution:* You describe a one-to-one correspondence; however, remember that yo list the elements in a table.)

11. **Diagonalization.** Cantor's proof is often referred to as "Cantor's diagonalization argument." Explain why this is a reasonable name.