My version of a solution to problem 4.19 in the 18th homework set:

#19 First, I will prove a lemma to use in the proof of the main problem.

Lemma: $1 + 2n \leq 2^n$ **for all** $n \in \mathbb{N}$ **such that** $n \geq 3$**.**

Proof of Lemma:

Let n $= 3$. Then $1 + 2n = 7 \leq 8 = 2^n$. Now suppose for some $w \in \mathbb{N}$ that $1 + 2w \leq 2^w$. We want to show that

$$1 + 2(w+1) \leq 2^{w+1}$$

.

Since $w \in \mathbb{N}$ it is clear that $w \geq 1/2$. By property B, $2w \geq 1$ and by property A, $2w + (2w+2) \geq 1 + (2w+2)$, or $4w + 2 \geq 2w + 3$. The induction hypothesis states that $1 + 2w \leq 2^w$, so again by B, we get that $2 + 4w \leq 2^{w+1}$. Property D implies then that

$$2^{w+1} \geq 2w + 3 = 2(w+1) + 1$$

which is what we wanted to show. Finally by the axiom of induction, $1 + 2n \leq 2^n$ for all $n \in \mathbb{N}$ such that $n \geq 3$.

Main Problem: Prove that $n^2 \leq 2^n + 1$**.**

Proof of main problem:

Let $n = 1$. Then $n^2 = 1 \leq 2^1 + 1 = 2^n + 1$. Let $n = 2$. Then $n^2 = 4 \leq 2^2 + 1 = 2^n + 1$. So the statement is true for $n = 1, 2$ and we need only show it is true for $n \geq 3$. We can do this by induction.

Let $n = 3$. Then $n^2 = 9 \leq 2^3 + 1 = 9 = 2^n + 1$. Now assume that for some $w \in \mathbb{N}$, $w \geq 3$, that $w^2 \leq 2^w + 1$. We need to show that this implies that $(w+1)^2 \leq 2^{w+1} + 1$.

By the lemma, we know that $1 + 2w \leq 2^w$. Since $2^w = 2^w \cdot 1 = 2^w(2-1) = 2^{w+1} - 2^w$, we have that

$$1 + 2w \leq 2^{w+1} - 2^w$$

and by property A

$$2^w + 1 + 2w \leq 2^{w+1} \ .$$

The induction hypothesis and property A imply that $w^2 + 2w \leq 2^w + 1 + 2w \leq 2^{w+1}$. So, by A and D, we have

$$(w+1)^2 = w^2 + 2w + 1 \leq 2^{w+1} + 1$$

which is what we wanted to show.

By the axiom of induction, $n^2 \leq 2^n + 1$ for all $n \geq 3$, and since we showed that $n^2 \leq 2^n + 1$ for $n = 1, 2$, we have that in fact $n^2 \leq 2^n + 1$ for all $n \in \mathbb{N}$.