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#9) Prove that if $f: X \to Y$ is 1-1 and $A, B \subseteq X$, then

a) $f(A \cap B) = f(A) \cap f(B)$.

Proof: Let $y \in f(A \cap B)$. Then by definition of $f(A \cap B)$, there is some $x \in A \cap B$ such that $f(x) = y$. Since $x \in A \cap B$ implies $x \in A$, so $f(x) = y$ implies $y \in f(A)$ by def. of $f(A)$. Since $x \in A \cap B$ implies $x \in B$, we have $f(x) = y$ implies $y \in f(B)$ by def of $f(B)$. Thus $y \in f(A) \cap f(B)$ by def. of intersection and $f(A \cap B) \subseteq f(A) \cap f(B)$

Now let $y \in f(A) \cap f(B)$. Thus $y \in f(A)$ and $y \in f(B)$ by def. of intersection. By def. of image, $\exists$ $x \in A$ and $z \in B$ such that $f(x) = y$ and $f(z) = y$. Thus $f(x) = f(z)$. But f is 1-1 so it must be that $x = z$. Since $x \in A$ and $x = z \in B$, $x \in A \cap B$. Now $f(x) = y$ implies $y \in f(A \cap B)$ by def. of $f(A \cap B)$. Thus we have $f(A \cap B) \supseteq f(A) \cap f$

Since $f(A \cap B) \subseteq f(A) \cap f(B)$ and $f(A \cap B) \supseteq f(A) \cap f(B)$, $f(A \cap B) = f(A) \cap f(B)$. //

b) $f(A \setminus B) = f(A) \setminus f(B)$.

Proof: Let $y \in f(A \setminus B)$. Then by def. of image there is some $x \in A \setminus B$ such that $f(x) = y$.
Since $x \in A \setminus B$ implies $x \in A$, we see that $f(x) = y$ implies $y \in f(A)$ by def. of image.
Now suppose that $\exists$ $b \in B$ such that $f(b) = y$.
Then $f(x) = f(b)$ and since f is 1-1, $x = b$.
This implies $x = b \in B$, but that contradicts the fact that $x \in A \setminus B$. So, there is no $b \in B$ such that $f(b) = y$, and $y \notin f(B)$.
Putting it all together, $y \in f(A)$, but $y \notin f(B)$, so by def. of compliment $y \in f(A) \setminus f(B)$, and
$f(A \setminus B) \subseteq f(A) \setminus f(B)$.

Let $y \in f(A) \setminus f(B)$. Then $y \in f(A)$ and $\exists$ some $x \in A$ such that $f(x) = y$. Suppose $x \in B$ as well. Then $f(x) = y$ implies that $y \in f(B)$ which contradicts the fact that $y \in f(A) \setminus f(B)$. So $x \notin B$, and $x \in A$, thus $x \in A \setminus B$ by def of compliment. So since $x \in A \setminus B$ and $f(x) = y$, $y \in f(A \setminus B)$ by def. of image, and $f(A) \setminus f(B) \subseteq f(A \setminus B)$.

Since $f(A \setminus B) \subseteq f(A) \setminus f(B)$ and $f(A \setminus B) \supseteq f(A) \setminus f(B)$, we have $f(A \setminus B) = f(A) \setminus f(B)$. //

c) $f^{-1}(f(A)) = A$.

<u>Proof</u>: Let $y \in f^{-1}(f(A))$. Then by def. of inverse image, ~~[illegible]~~ $f(y) \in f(A)$. By def. of image, there exists some $x \in A$ such that $f(x) = f(y)$. Since f is 1-1, we have $y = x \in A$, so $f^{-1}(f(A)) \subseteq A$.

Now let $y \in A$. Since f is a function, $f: X \to Y$, and $A \subseteq X$, there is some $x \in Y$ such that $f(y) = x$. Now, $y \in A$ and $f(y) = x$ implies that $x \in f(A)$, so more precisely $f(y) = x \in f(A)$. Thus, by def. of inverse image $y \in f^{-1}(f(A))$, and $A \subseteq f^{-1}(f(A))$.

Since $A \subseteq f^{-1}(f(A))$ and $A \supseteq f^{-1}(f(A))$, we have $A = f^{-1}(f(A))$.

#3.10 : Suppose $f: A \to B$.

a) Prove that if f is 1-1 from A onto $f(A)$, then f^{-1} is 1-1 from $f(A)$ onto A.

Proof: We want to show that $f^{-1}: f(A) \to A$ is 1-1 and onto. Note f^{-1} exists because f is 1-1 & onto by Thm 3.24.

1-1: Let $x, y \in f(A)$, and suppose that $f^{-1}(x) = f^{-1}(y)$. If we apply f to both the sides,

$$x = f(f^{-1}(x)) = f(f^{-1}(y)) = y.$$

by fact that f^{-1} is the inverse function for f.

by the fact that f is a function

So, if $f^{-1}(x) = f^{-1}(y)$ for some $x, y \in f(A)$ then $x = y$, and f^{-1} is 1-1.

onto: Let $a \in A$. Since f is a function $f(a) = b$ for some $b \in B$. In fact, since $b = f(a)$ and $a \in A$, $b \in f(A)$ by the definition of image. Now, f^{-1} is a function on $f(A)$, so

$$f^{-1}(b) = y \text{ for some } y \in A, \text{ and}$$

more precisely since $b = f(a)$,

$$f^{-1}(b) = f^{-1}(f(a)) = a \in A$$

by the fact that f^{-1} is the inverse function for f.

So given any $a \in A$, we can find $b = f(a) \in f(A)$ such that $f^{-1}(b) = a$, and f^{-1} maps $f(A)$ onto A.

3.10: b) Prove that if f is 1-1 from A onto B, then f^{-1} is 1-1 from B onto A.

Proof: Again $f: A \to B$ is 1-1 and onto implies that there exists $f^{-1}: B \to A$ by Thm 3.24. We want to show that f^{-1} is 1-1 and onto.

1-1: Let $b, c \in B$, and suppose $f^{-1}(b) = f^{-1}(c)$. Then

$$b = f(f^{-1}(b)) = f(f^{-1}(c)) = c .$$

by def. of inverse function

by fact that f is a function

Thus, f^{-1} is 1-1 from B into A.

onto: take any $a \in A$. Since f is a function such that $f: A \to B$, there is some $b \in B$ such that $f(a) = b$. So $f^{-1}(f(a)) = f^{-1}(b) = a$ and f^{-1} is onto.