Solutions to homework – Section 3.3:

1. He showed that the real numbers and the natural numbers have different cardinalities, so that there is more than one "size" of infinity.

2. $0.\underline{1}2345$ ← 1^{st} digit is 1
   $0.2\underline{4}242$ ← 2^{nd} digit is 4
   $0.98\underline{7}65$ ← 3^{rd} digit is 7

   (by "first digit", generally mean "first nonzero digit")

3. It's constructed by listing all the natural numbers consecutively.

   $0.123456789\underset{\uparrow 10^{th}}{1}011\underset{\uparrow 14^{th}}{\underline{1}}2131415161\underset{\uparrow 25^{th}}{\underline{7}}18192\underset{\uparrow 31^{st}}{\underline{0}}\ldots$

4. Any 5-digit number whose first digit is not 3, second is not 8, and third digit is not 2 will definately not be in the list →
   for example, 49311 cannot be listed.

5. No, you cannot, because no matter what number I give you, you cannot know whether it's equal to the last number on the list or not.

9. Suppose we have a listing of all the real numbers, as Cantor did. We are going to write down a number, called M, that is missing from this list. We can suppose that $M = 0.???\ldots$ (so it is some number between 0 and 1). Each digit of M will either be a 4 or an 8, and we'll decide which one each digit is by the following:
   For the 1^{st} digit of M, we look at the number in the list that is first. Either it is a 4 or it is not a 4. If the 1^{st} digit of the 1^{st} # of the list is a 4, then we make the 1^{st} digit of M to be 8.

If it is <u>not</u> 4, we set the first digit of M to be 4. Now, for the second digit of M, we look at the 2nd digit of the 2nd number in the list. Again, if it's 4, then the 2nd digit of M is set to 8, and if it's <u>not</u> 4, then the second digit of M is 4. We continue on in this way, so that the n^{th} digit of M is 4 <u>if</u> the n^{th} digit of the n^{th} number in the list is <u>not</u> 4, and is 8 if the n^{th} digit of the n^{th} number in the list <u>is</u> 4.

By the way we have created M, it cannot be equal to <u>any</u> number in the list! We can do this process regardless of what list we begin with, so the only conclusion is that there is no possible way to list <u>all</u> of the real numbers. This tells us that there is no one-to-one correspondence between the real numbers + the natural numbers, since <u>if</u> <u>there</u> <u>was</u> a 1-1 corresp., we would then be able to list all the reals.

11. diagonalization is a nice name for the process of Cantor's proof because as you construct the number M that's missing from the list, you only need look at the 1st digit of the 1st number, 2nd digit of the 2nd number, etc, which creates a diagonal in the list. For example:

$$\begin{array}{l}
1 \leftrightarrow 0.5967784123\ldots \\
2 \leftrightarrow 0.4125689712\ldots \\
3 \leftrightarrow 0.5432142612\ldots \\
4 \leftrightarrow 0.0041002100\ldots \\
\vdots
\end{array}$$

14. Each flip gives some infinite sequence of H's and T's

for example:

Person:	1	2	3	4	5	6	....
Result:	H	T	H	H	H	T	....

So the set of all possible outcomes would be the set of all possible infinite lists of H's and T's.

We could show that there is no 1-1 correspondence between the set of all outcomes & $\mathbb{N}$ by the same diagonalization technique Cantor used:

Suppose there is a 1-1 correspondence

1 ⟷ HTHHHTH...

2 ⟷ HHTHHTTH...

3 ⟷ THHHTHHT...

4 ⟷ THTHTHT...

⋮

Then we can always create an outcome that's missed by changing the entry on the diagonal:

missed outcome = THHH ...

This contradicts the claim that we had a 1-1 correspondence.

16. Show set of all real numbers between 0 and 1 just having 1's and 2's after the decimal has a greater cardinality than the natural numbers ($\mathbb{N}$).

Again, we can show this by contradiction.
Assume they have the <u>same</u> cardinality. Then I can list <u>all</u> the numbers between 0 and 1 whose digits are only 1's and 2's. For example:

$$1 \leftrightarrow 0.\underline{1}221112121\ldots$$
$$2 \leftrightarrow 0.1\underline{2}1212121\ldots$$
$$3 \leftrightarrow 0.11\underline{1}12222121l\ldots$$
$$4 \leftrightarrow 0.122\underline{2}11121211122 11\ldots$$
$$\vdots$$

Now, I can always construct <u>another</u> number between 0 and 1 whose digits are only 1's and 2's by again using diagonalization: If the n^{th} digit of the n^{th} number in the list is a 1, I can make the n^{th} digit of the missing number a 2 and vice-versa.
So for the list above, my missing number is

$$m = 0.2121\ldots$$

This contradicts the claim that I had a 1-1 correspondence with $\mathbb{N}$.
<u>So</u>, $\mathbb{N}$ and the set of all reals between 0 and 1 with digits only 1's and 2's do <u>not</u> have the same cardinality.

19. Since the Hotel Cardinality only has as many rooms as there are natural numbers, if we had a group of people whose cardinality was the same as the real numbers, there would be no way the Hotel manager could give everyone a room, because there is no 1-1 corresp. between $\mathbb{N}$ (nat. numbers) and $\mathbb{R}$ (real numbers).

(