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Math 231 : Introduction to Ordinary Differential Equations
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Today we look at extensions of three methods we've already learned.

1 Almost Exact Equations

First , an "almost exact" equation is an ode of the form

$$M(x, y)dx + N(x, y)dy = 0$$

that is not exact, but that becomes exact when multiplied by some function $f(x, y)$.

We cannot *always* make an equation exact, but there are some equations we are guaranteed are almost exact.
How do we know when an equation is almost exact? And how do we find what to multiply it by to make it exact?

Key Result: If $\frac{\frac{\delta M}{\delta y} - \frac{\delta N}{\delta x}}{N}$ can be reduced to a function only of x, then the equation is almost exact, and multiplying it by

$$e^{\int \frac{\frac{\delta M}{\delta y} - \frac{\delta N}{\delta x}}{N} dx}$$

will make it exact.

Likewise, if $\frac{\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}}{M}$ can be reduced to a function only of y, then the equation is almost exact, and multiplying it by

$$e^{\int \frac{\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}}{M} dy}$$

will make it exact.

Example 1. Solve

$$(2xy^2 + \frac{y}{x})\,dx + (x^2y + ye^y)\,dy = 0\,.$$

Solution: Exact? $\frac{\partial M}{\partial y} = 4xy + \frac{1}{x}$, $\frac{\partial N}{\partial x} = 2xy \Rightarrow$ No

Almost exact? (1) $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{2xy + \frac{1}{x}}{y(x^2+e^y)}$ ← not dependent on only x, so this is not helpful.

(2) $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \dfrac{-2xy - \frac{1}{x}}{2xy^2 + \frac{y}{x}} = \dfrac{-(2xy+\frac{1}{x})}{y(2xy+\frac{1}{x})} = \boxed{-\frac{1}{y}}$ Yes! depends only on y!

$\Rightarrow \mu = e^{\int -\frac{1}{y}dy} = y^{-1} \Rightarrow$ eqn becomes

$(2xy + \frac{1}{x})dx + (x^2 + e^y)dy = 0$ (if $y \neq 0$)

which is exact, so

$\frac{\partial f}{\partial x} = 2xy + \frac{1}{x}$ and $\frac{\partial f}{\partial y} = x^2 + e^y$

$\Rightarrow f(x,y) = \int 2xy + \frac{1}{x}\,dx = x^2y + \ln|x| + g(y)$

$\Rightarrow \frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 + e^y \Rightarrow g(y) = e^y.$

Gen sol'n:

$$\boxed{x^2y + \ln|x| + e^y = C \quad \text{or } y = 0}$$

Example 2. Solve

$$(2y^3 + 2y^2)\,dx + (3y^2x + 2xy)\,dy = 0$$

Solution: Exact? $\frac{\partial M}{\partial y} = 6y^2 + 4y$ $\frac{\partial N}{\partial x} = 3y^2 + 2y \Rightarrow$ NO.

Almost exact? $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{3y^2+2y}{3y^2x + 2xy} = \dfrac{1}{x}$ ← depends only on x, so yes!

$\mu = e^{\int \frac{1}{x}dx} = x$, so if $x \neq 0$, eq'n is equivalent to:

$(2y^3x + 2y^2x)\,dx + (3y^2x^2 + 2x^2y)\,dy = 0$

which is exact. $\frac{\partial f}{\partial x} = 2y^3x + 2y^2x$ and $\frac{\partial f}{\partial y} = 3y^2x^2 + 2x^2y$.

So $f(x,y) = y^3x^2 + y^2x^2 + g(y) \Rightarrow \frac{\partial f}{\partial y} = 3y^2x^2 + 2yx^2 + g'(y)$.

$\Rightarrow g'(y) = 0$, so $g(y) = C$. Gen sol'n is:

$$\boxed{y^3x^2 + y^2x^2 = C \quad \text{or } x = 0}$$

2 Bernoulli Equations

Next, we'll look at a type of equation that through a change of variables can always be transformed into a linear equation.

A first order o.d.e. is Bernoulli if it has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)y^n \ .$$

Let's first look at an example before we talk about the general method for solving Bernoulli equations:

Example 3. Solve

$$x\frac{dy}{dx} + y = 3xy^3 \ .$$

Solution:

$$\frac{dy}{dx} + \frac{1}{x}y = 3y^3$$

$$\Rightarrow y^{-3}\frac{dy}{dx} + \frac{1}{x}y^{-2} = 3 \quad (\text{if } y \not\equiv 0)$$

Let $v = y^{-2} \Rightarrow \frac{dv}{dx} = -2y^{-3}\frac{dy}{dx} \Rightarrow$ eq'n becomes: $\boxed{-\frac{1}{2}\frac{dv}{dx} + \frac{1}{x}v = 3}$

Linear!

So: $\frac{dv}{dx} - \frac{2}{x}v = -6$ and $\mu = e^{\int -\frac{2}{x}dx} = \underline{x^{-2}}.$

$$\Rightarrow \frac{d}{dx}(x^{-2}v) = -6x^{-2} \Rightarrow x^{-2}v = +6x^{-1} + C$$

$$\Rightarrow v = 6x + Cx^2 \Rightarrow \boxed{\begin{array}{l} y^{-2} = 6x + Cx^2. \\ \text{or } y \equiv 0 \end{array}}$$

General Method We can carry out analogous steps for *any* Bernoulli o.d.e. and find it's general solution.

Bernoulli has form $\quad a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)y^n.$

1. Put in standard form: $\quad \frac{dy}{dx} + P(x)y = Q(x)y^n.$

2. $\div$ by y^n: $\quad$ If $y \neq 0$, $\quad y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x).$

3. Let $v = y^{1-n}$, then $\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$ and the eq'n is: $\boxed{\frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x)}$ which is <u>linear</u>.

4. Solve as linear and substitute $v = y^{1-n}$ in end result. Acknowledge that $y \equiv 0$ is also a sol'n.

Example 4. Solve

$$\frac{dy}{dx} + y = e^{3x}y^5 .$$

If $y \neq 0 \Rightarrow y^{-5}\frac{dy}{dx} + y^{-4} = e^{3x}$. $\quad$ Let $v = y^{-4} \Rightarrow \frac{dv}{dx} = -4y^{-5}\frac{dy}{dx}$

$$\Rightarrow -\frac{1}{4}\frac{dv}{dx} + v = e^{3x} \Rightarrow \frac{dv}{dx} - 4v = -4e^{3x}.$$

<u>Solve as linear:</u> $\mu = e^{-4x} \Rightarrow \frac{d}{dx}\left(e^{-4x}v\right) = -4e^{-x}.$

$$\Rightarrow e^{-4x}v = 4e^{-x} + C \Rightarrow v = 4e^{3x} + Ce^{4x}$$

$\Rightarrow \boxed{y^{-4} = 4e^{3x} + Ce^{4x} \quad \text{or } y \equiv 0}$ $\leftarrow$ general sol'n.

3 Homogeneous Equations

It is important to point out here that we'll be using the term "homogeneous equation" in **two different ways** during this course. For now, we define a homogeneous first order equation to be any equation of the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

As we did for the Bernoulli equations, we will solve these by making a change of variables. All Bernoulli equations can be transformed to be **separable** through a specific change of variables.

Again, we'll look at an example before we talk about the general method for solving first order homogeneous equations.

Example 5. Solve

$$\frac{dy}{dx} = \frac{x-y}{x+y}.$$

Solution:

Put in homog. form: $\frac{dy}{dx} = \frac{(x-y)\cdot(\frac{1}{x})}{(x+y)\cdot(\frac{1}{x})} = \frac{(1-y/x)}{(1+y/x)}$.

Let $v = y/x$. Then $y = xv \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$.

$\Rightarrow$ $v + x\frac{dv}{dx} = \frac{1-v}{1+v}$ ← is separable.

$$x\frac{dv}{dx} = \frac{1-v}{1+v} - v = \frac{1-v-v-v^2}{1+v} = \frac{1-2v-v^2}{1+v}$$

(If $1-2v-v^2 \neq 0$) $\Rightarrow \int \frac{1+v}{1-2v-v^2}\,dv = \int \frac{1}{x}\,dx$. Let $u = 1-2v-v^2$ $\Rightarrow du = (-2-2v)dv = -2(1+v)dv$

↓

or $1-\frac{2y}{x}-\frac{y^2}{x^2} \neq 0$

↓

$x^2-2yx-y^2 \neq 0$.

$\Rightarrow -\frac{1}{2}\int \frac{1}{u}\,du = \ln|x| + C \Rightarrow -\frac{1}{2}\ln\left|1-\frac{2y}{x}-\frac{y^2}{x^2}\right| = \ln|x| + C$

Is $x^2-2yx-y^2=0$ a sol'n? $\Rightarrow 2x-2y-2x\frac{dy}{dx}-2y\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}(-2x-2y) = 2y-2x \Rightarrow \frac{dy}{dx} = \frac{x-y}{x+y} \Rightarrow$ yes!

$$\boxed{\text{Gen sol'n: } x^2-2yx-y^2=0 \ \text{ or } \ -\tfrac{1}{2}\ln\left|1-\tfrac{2y}{x}-\tfrac{y^2}{x^2}\right| = \ln|x|+C}$$

General Method We can carry out analogous steps for *any* first order homogeneous o.d.e. and find it's general solution.

1. Put in form $\frac{dy}{dx} = f(y/x)$.

2. let $v = y/x \Rightarrow v + x\frac{dv}{dx} = \frac{dy}{dx}$, and rewrite eqn in terms of v and x.

3. Solve as <u>separable</u> $v + x\frac{dv}{dx} = f(v)$

$$\Rightarrow \int \frac{1}{f(v)-v}\,dv = \int \frac{1}{x}\,dx \quad (\text{if } f(v)-v \neq 0)$$

4. Replace $v = y/x$ in general solution.

Example 6. Solve

$$(xy + y^2 + x^2)\,dx - x^2\,dy = 0\,.$$

Solution:

If $x \neq 0$: $\frac{dy}{dx} = \frac{xy + y^2 + x^2}{x^2} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 + 1$

Let $v = (y/x) \Rightarrow x\frac{dv}{dx} + v = \frac{dy}{dx}$ so ode is:

$$v + x\frac{dv}{dx} = v + v^2 + 1$$

$\Rightarrow x\frac{dv}{dx} = v^2 + 1 \Rightarrow \int \frac{1}{v^2+1}\,dv = \int \frac{1}{x}\,dx \Rightarrow \arctan(v) = \ln|x| + c$

$$\Rightarrow \boxed{\begin{array}{c}\arctan(y/x) = \ln|x| + C \\ \text{or } x = 0\end{array}}$$