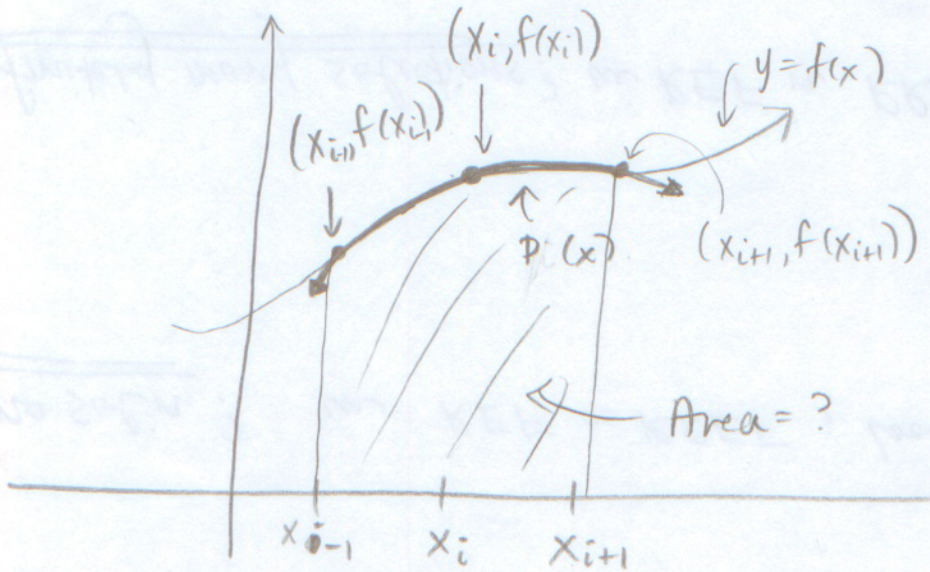
To derive Simpson's Rule, we essentially need to find the area under a quadratic polynomial determined by 3 points spaced by Δx apart:



$P_i(x) = Cx^2 + Dx + E$ for some C, D, & E.

and we need it to pass through $(x_{i-1}, f(x_{i-1}))$, $(x_i, f(x_i))$ and $(x_{i+1}, f(x_{i+1}))$: so:

$$\text{need: } \begin{cases} P(x_{i-1}) = C(x_{i-1})^2 + Dx_{i-1} + E = f(x_{i-1}) \\ P(x_i) = C(x_i)^2 + Dx_i + E = f(x_i) \leftarrow (*) \\ P(x_{i+1}) = C(x_{i+1})^2 + Dx_{i+1} + E = f(x_{i+1}) \end{cases}$$

Since $x_{i-1} = x_i - \Delta x$ and $x_{i+1} = x_i + \Delta x$:

$$f(x_{i-1}) + f(x_{i+1}) = C[x_i - \Delta x]^2 + D(x_i - \cancel{\Delta x}) + E + C(x_i + \Delta x)^2 + D(x_i + \cancel{\Delta x}) + E$$

$$= C(x_i^2 - \cancel{2\Delta x\, x_i} + \Delta x^2) + 2Dx_i + 2E + C(x_i^2 + \cancel{2\Delta x\, x_i} + \Delta x^2)$$

$$= 2(Cx_i^2 + Dx_i + E) + 2C\Delta x^2 = 2f(x_i) + 2C\Delta x^2 \quad (*2)$$

And the area will be

$$\int_{x_{i-1}}^{x_{i+1}} P(x)\,dx = \int_{x_{i-1}}^{x_{i+1}} Cx^2 + Dx + E\,dx = \frac{C}{3}x^3 + \frac{D}{2}x^2 + Ex \Big|_{x_{i-1}}^{x_{i+1}}$$

$$= \frac{C}{3}\left((x_{i+1})^3 - (x_{i-1})^3\right) + \frac{D}{2}\left((x_{i+1})^2 - (x_{i-1})^2\right) + E\left(x_{i+1} - x_{i-1}\right)$$

since: $x_{i+1} = x_i + \Delta x$; $x_{i-1} = x_i - \Delta x$ $\Rightarrow$

$$= \frac{C}{3}\left[(x_i + \Delta x)^3 - (x_i - \Delta x)^3\right] + \frac{D}{2}\left[(x_i + \Delta x)^2 - (x_i - \Delta x)^2\right] + E \cdot 2\Delta x$$

$$= \frac{C}{3}\left[\left(\cancel{x_i^3} + 3x_i^2\Delta x + \cancel{3x_i\Delta x^2} + \Delta x^3\right) - \left(\cancel{x_i^3} - 3x_i^2\Delta x + \cancel{3x_i\Delta x^2} - \Delta x^3\right)\right]$$

$$+ \frac{D}{2}\left[\left(\cancel{x_i^2} + 2x_i\Delta x + \cancel{\Delta x^2}\right) - \left(\cancel{x_i^2} - 2x_i\Delta x + \cancel{\Delta x^2}\right)\right] + 2E\Delta x$$

$$= \frac{C}{3}\left(6x_i^2\Delta x + 2\Delta x^3\right) + \frac{D}{2}\left(4x_i\Delta x\right) + 2E\Delta x$$

$$= 2Cx_i^2\Delta x + 2Dx_i\Delta x + 2E\Delta x + \frac{2C}{3}\Delta x^3$$

$$= 2\Delta x\left[Cx_i^2 + Dx_i + E\right] + \frac{2C}{3}(\Delta x)^3$$

by (*) $\rightarrow$ $= 2\Delta x\, f(x_i) + \frac{2C}{3}(\Delta x)^3$

Since by (*2) we know $f(x_{i+1}) + f(x_{i-1}) = 2f(x_i) + 2C(\Delta x)^2$

we have $\Rightarrow$ $2C(\Delta x)^2 = \left(f(x_{i+1}) + f(x_{i-1}) - 2f(x_i)\right)$ Δx

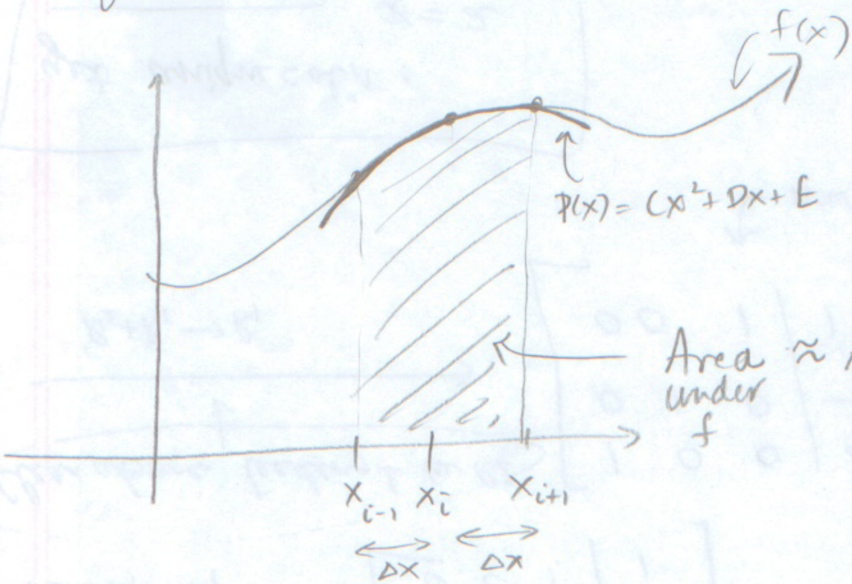
$$= 2\Delta x f(x_i) + \frac{\Delta x}{3}\left(f(x_{i+1}) + f(x_{i-1}) - 2f(x_i)\right) \Rightarrow \frac{2C(\Delta x)^3}{3} = \left(f(x_{i+1}) + f(x_{i-1}) - 2f(x_i)\right)\frac{\Delta x}{3}$$

$$= \frac{\Delta x}{3}\left[f(x_{i+1}) + f(x_{i-1}) - 2f(x_i) + 6f(x_i)\right] = \frac{\Delta x}{3}\left[f(x_{i-1}) + 4f(x_i) + f(x_{i+1})\right]$$

So the final result is that

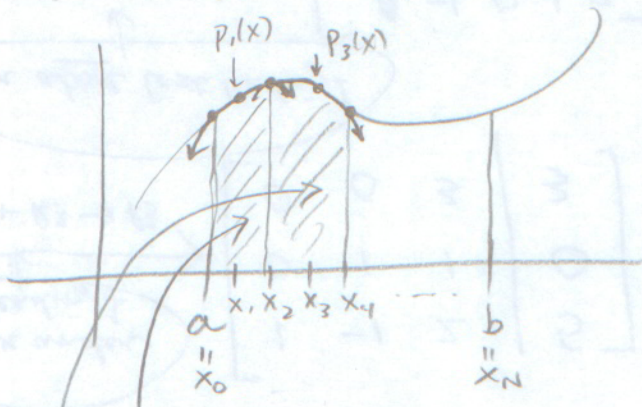
$$\int_{x_{i-1}}^{x_{i+1}} P(x)\,dx = \frac{\Delta x}{3}\left[f(x_{i-1}) + 4f(x_i) + f(x_{i+1})\right]$$

gives the area under the quatratic over $[x_{i-1}, x_{i+1}]$.



Area under f $\approx$ Area under P $= \frac{\Delta x}{3}\left[f(x_{i-1}) + 4f(x_i) + f(x_{i+1})\right]$

So for simpson's rule, when we have multiple subintervals,



$\text{Area}_1 \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$

$\text{Area}_2 \approx \frac{\Delta x}{3}\left[f(x_2) + 4f(x_3) + f(x_4)\right]$

etc...

adding up these areas gives $\displaystyle\int_a^b f(x)\,dx \approx \overbrace{\frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{N-2}) + 4f(x_{N-1}) + f(x_N)\right]}^{\text{Simpson's Rule!}}$