Homework Set # 6 – Math 435 – Summer

1. Solve $u_{xx} + u_{yy} = 0$ on the rectangle $0 < x < 2$, $0 < y < 3$, with boundary conditions

$$u_x(0, y) = 0$$
$$u_x(2, y) = 0$$
$$u(x, 0) = 0$$
$$u(x, 3) = 3x$$

Solution:

We can seperate variables to obtain $u(x, y) = X(x)Y(y)$ where $X'' = \lambda X$ and $Y'' = -\lambda Y$, and we have boundary conditions $X'(0) = X'(2) = 0$ and $Y(0) = 0$. Solving the equation for X, we consider the three cases:

$\lambda = 0$ Then $X(x) = c_1x + c_2$. Since $X' = c_1$ the b.c.'s imply that $c_1 = 0$, so we get that

$$X = c_2$$

for any constant is possible.

$\lambda > 0$ Then $\lambda = \beta^2$ for some $\beta > 0$. This yields the general solution $X = c_1e^{\beta x} + c_2e^{-\beta x}$. Then $X' = \beta(c_1e^{\beta x} - c_2e^{-\beta x})$. Applying the b.c.'s gives $c_1 = c_2$ and $\beta c_1(e^{2\beta} - e^{-2\beta}) = 0$. The only way this can be true is if $c_1 = 0$ which yields the trivial solution $X = 0$.

$\lambda < 0$ Now, we can write $\lambda = -\beta^2$ for some $\beta > 0$. This yield the general (real) solution $X = c_1 \cos(\beta x) + c_2 \sin(\beta x)$. Thus $X' = \beta(-c_1 \sin(\beta x) + c_2 \cos(\beta x))$. Applying the b.c.'s gives $c_2 = 0$ and $2\beta = n\pi$ or $\beta = \frac{n\pi}{2}$. Thus

$$X(x) = C_1 \cos\left(\frac{n\pi x}{2}\right) .$$

Now we need to find the corresponding solutions Y.

$\lambda = 0$ Thus $Y(y) = c_1y + c_2$. Applying the BC $Y(0) = 0$ gives that $c_2 = 0$ or $Y(y) = c_1y$. So the solution $u(x, y)$ corresponding to $\lambda = 0$ is

$$u_0(x, y) = Cy$$

$\lambda < 0$ Recall from above that this implies that $\lambda = -\frac{n^2\pi^2}{4}$. Thus we want to solve $Y'' = \frac{n^2\pi^2}{4}Y$, which has general solution $Y = c_1 \cosh\left(\frac{n\pi y}{2}\right) + c_2 \sinh\left(\frac{n\pi y}{2}\right)$. Applying the BC $Y(0) = 0$, we get $c_1 = 0$, so

$$Y = c \sinh\left(\frac{n\pi y}{2}\right) .$$

Our corresponding solutions $u_n(x, y)$ are then

$$u_n(x, y) = c_n \cos\left(\frac{n\pi x}{2}\right) \sinh\left(\frac{n\pi y}{2}\right) .$$

Finally, we get our full solution by the superposition principle

$$u(x, y) = C_0y + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{2}\right) \sinh\left(\frac{n\pi y}{2}\right) .$$

We can finally impose the final boundary condition that $u(x, 3) = 3x$:

$$(x, 3) = 3C_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{2}\right) \sinh\left(\frac{3n\pi}{2}\right) = 3x\ .$$

Using Fourier coefficients we have

$$C_0 = \frac{1}{6} A_0 = \frac{1}{6} \int_0^2 3x\, dx = \frac{1}{4} x^2 |_0^2 = 1$$

and

$$\sinh\left(\frac{3n\pi}{2}\right) c_n = \int_0^2 3x \cos(n\pi x/2)\, dx = \frac{6x}{n\pi} \sin(n\pi x/2)|_0^2 - \int_0^2 \frac{6}{n\pi} \sin(n\pi x/2)\, dx$$

continuing we see

$$\sinh\left(\frac{3n\pi}{2}\right) c_n = \frac{12}{n^2\pi^2} \cos(n\pi x/2)|_0^2 = \frac{12}{n^2\pi^2} \left((-1)^n - 1\right)\ .$$

2. Consider Laplace's equation on the disk $x^2 + y^2 < 9$ with boundary condition $u(x, y) = \frac{x^2}{3}$ when $x^2 + y^2 = 9$.

   (a) Without solving the equation, give the value of u at the center of the disk.

   (b) Solve the equation.

   Solution:

   (a) Using the Mean Value Principle, we know that u at the center is equal to the average value of u around the boundary, so in polar coordinates we find

   $$u(0, \theta) = \frac{1}{2\pi} \int_0^{2\pi} 3\cos^2(\theta) d\theta$$

   Solving by using the identity $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$, we get

   $$u(0, \theta) = \frac{3}{2}\ .$$

   (NOTE: we could also find the boundary integral as a line integral. In this case we parametrize the boundary curve by $s(\theta) =< 3\cos\theta, 3\sin\theta >$ wtih $0 \leq \theta \leq 2\pi$. Then

   $$\int_{\partial D} h\, dS = \int_0^{2\pi} h(s(\theta))|s'(\theta)| d\theta = \int_0^{2\pi} 3\cos^2(\theta) 3\, d\theta$$

   and the average value is the value of this integral divided by the circumference $C = 6\pi$ which yields the same result.)

   (b) If we follow the seperation of variables process as in class (or the book) for polar coordinates, and rule out any part of the solution that goes to infinity at $r = 0$, we have

   $$u(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^n (d_n \cos(n\theta) + e_n \sin(n\theta))\ .$$

Applying our boundary condition gives

$$u(3,\theta) = C_0 + \sum_{n=1}^{\infty} 3^n (d_n \cos(n\theta) + e_n \sin(n\theta) = 3\cos^2(\theta) = \frac{3 + 3\cos(2\theta)}{2} \ .$$

Now, we can see that $C_0 = \frac{3}{2}$ and $9d_2 = \frac{3}{2}$ or $d_2 = \frac{1}{6}$, while $d_n = 0$ for all $n \neq 2$ and $e_n = 0$ for all n. Thus the solution is

$$u(r,\theta) = \frac{3}{2} + \frac{1}{6} r^2 \cos(2\theta) \ .$$

3. Use separation of variables to derive the solution $u(r,\theta)$ to Laplace's equation on the annulus $1 < r < 3$ with boundary conditions:

$$u(1,\theta) = \sin^2(\theta)$$
$$u(3,\theta) = 0$$

Solution: Again, follow the derivation via separation of variables we did in class (or in the book), only this time, since $r = 0$ is NOT included in our domain, we arrive at solutions of the form

$$u(r,\theta) = C_0 + C_1 \ln r + \sum_{n=1}^{\infty} (a_n r^n + b_n r^{-n})(d_n \cos(n\theta) + e_n \sin(n\theta) \ .$$

This time, $r = 0$ is not part of our domain, so we need not worry about the behavoir of our solution at $r = 0$. If we first apply the boundary condition at $r = 3$, we see

$$u(3,\theta) = C_0 + C_1 \ln 3 + \sum_{n=1}^{\infty} (a_n 3^n + b_n 3^{-n})(d_n \cos(n\theta) + e_n \sin(n\theta) = 0 \ .$$

Thus, $C_0 + C_1 \ln 3 = 0$ and $a_n 3^n + b_n 3^{-n} = 0$ for all n. We can then rewrite our solution accordingly as

$$u(r,\theta) = C_0 (1 - \frac{\ln r}{\ln 3}) + \sum_{n=1}^{\infty} (r^n - 3^{2n} r^{-n})(d_n \cos(n\theta) + e_n \sin(n\theta) \ .$$

Now, applying the b.c. at $r = 1$ we have

$$u(r,\theta) = C_0 + \sum_{n=1}^{\infty} (1 - 3^{2n})(d_n \cos(n\theta) + e_n \sin(n\theta) = \sin^2(\theta) = \frac{1 - \cos(2\theta)}{2} \ .$$

By sight, we can see that $C_0 = \frac{1}{2}$ and $d_2 = \frac{-1}{2(1-3^{2n})}$, while $d_n = 0$ if $n \neq 2$ and $e_n = 0$ for all n. This gives a solution of

$$u(r,\theta) = \frac{1}{2}\left(1 - \frac{\ln r}{\ln 3}\right) + \frac{r^2 - 81 r^{-2}}{160} \cos(2\theta) \ .$$

4. Consider the steady-state temperature distribution inside a spherical ball ($r < 4$), whose outer boundary sphere is held at a constant temperature of 10 degrees. According to the maximum

principle for Laplace's equation in 3D, what can you conclude about $u(x, y, z)$ inside the ball? What does the minimum principle tell you?

solution The maximum principle says that $u(x, y, z) \leq 10$ for all (x, y, z) in the interior of the ball, and the minimum principle says that$u(x, y, z) \geq 10$ for all (x, y, z) in the interior of the ball. The maximum and minimum principles together tell us that $u(x, y, z) = 10$ for all points (x, y, z) in the ball.

5. Problem 5 from the exercises for section 7.1 in Strauss.

Solution:

Suppose that u is harmonic and $\frac{\partial u}{\partial n} = h$ on ∂D and that w is **any** real-valued function on D. Let $v = w - u$, or $w = u + v$. Subbing v into E, we get

$$E[w] = \frac{1}{2} \int\int\int_D |\nabla w|^2 \, dx - \int\int_{\partial D} h[w] \, dS \ .$$

Now, expanding $w = u + v$, we have

$$E[w] = \frac{1}{2} \int\int\int_D \left(|\nabla u|^2 + |\nabla v|^2 + 2\nabla v \cdot \nabla u \right) \, dx - \int\int_{\partial D} h[u] \, dS - \int\int_{\partial D} h[v] \, dS \ .$$

By Green's first identity for u and v,

$$\int\int\int_D \nabla v \cdot \nabla u \, dx = \int\int_{\partial D} v \frac{\partial u}{\partial n} \, dS - \int\int\int_D v \Delta u \, dx$$

and since $\frac{\partial u}{\partial n} = h$ on the boundary,

$$\int\int\int_D \nabla v \cdot \nabla u \, dx = \int\int_{\partial D} vh \, dS \ .$$

Subbing this in above, we have

$$E[w] = \frac{1}{2} \int\int\int_D |\nabla v|^2 \, dx + \left(\frac{1}{2} \int\int\int_D |\nabla u|^2 - \int\int_{\partial D} uh \, dS \right)$$

or

$$E[w] = \frac{1}{2} \int\int\int_D |\nabla v|^2 \, dx + E[u] \ .$$

This implies that

$$E[w] \geq E[u]$$

and in fact that if $u \neq w$,

$$E[w] > E[u] \ .$$

Thus, the harmonic function satisfying the given Neumann boundary condition is the function that minimizes E_h over all real-valued functions on D.

6. Show that the Green's function is unique for a given domain. (hint: take the difference of two of them and use the proof that solutions to Laplace's equation with Dirichlet b.c.'s are unique)

Solution:

Recall that the solution to Laplace's equation with Dirichlet boundary conditions is unique. Suppose that we have two different green's functions on D: $G_1(x, x_0)$ and $G_2(x, x_0)$. Then by the definition of a Green's function we have harmonic functions H_1 and H_2 such that $G_1 = v + H_1$ and $G_2 = v + H_2$ where $v = \frac{-1}{4\pi|x-x_0|}$. Notice then that if we let $g = G_1 - G_2 = H_1 - H_2$, then we get that $\Delta g = 0$ on all of D and $g = 0$ on ∂D. By the fact that solutions to Laplace's equation with Dirichlet b.c.'s are unique, it must be that $g = 0$ on all of D. This implies that $G_1 = G_2$.

7. The Neumann function $N(x, y)$ for a domain D is defined exactly like the Green's function in Section 7.3 except that (ii) is replaced by the Neumann boundary condition

$$\frac{\partial N}{\partial n} = 0$$

for $x \in \partial D$. In this case, we get the analogous statement to Theorem 7.3.1: If $N(x, x_0)$ is the Neumann function, then the solution of the Dirichlet problem is given by the formula

$$u(x_0) = -\int\int_{\partial D} N(x, x_0)\frac{\partial u}{\partial n}\, dS\ .$$

Show this is true.

Proof:

The representation formula for harmonic functions is

$$u(x_0) = \int\int_{\partial D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\, dS$$

where $v(x) = \frac{-1}{4\pi|x-x_0|}$. Writing $N(x, x_0) = v + H$, where H is harmonic in all of D, and applying Green's 2nd identity, we have

$$0 = \int\int_{\partial D} \left(u\frac{\partial H}{\partial n} - H\frac{\partial u}{\partial n}\right)\, dS\ .$$

Adding our two expressions together, we have

$$u(x_0) = \int\int_{\partial D} \left(u\frac{\partial N}{\partial n} - N\frac{\partial u}{\partial n}\right)\, dS\ .$$

Now, since we are requiring that $\frac{\partial N}{\partial n} = 0$ on the boundary, we get

$$u(x_0) = -\int\int_{\partial D} N\frac{\partial u}{\partial n}\, dS\ .$$

8. Use Green's functions to solve

$$\Delta u = 0$$

inside the ball of radius $r = 3$, if we assume that $u = \frac{x+y}{3}$ on the boundary of the ball (the sphere of radius 3). Notice that the Green's function expression on a sphere that we talked about in class cannot be used to find the value of u at the origin (why?). How **could** we find the value of u at the origin? (note: you don't actually have to find the value, just describe how to)