Homework Set # 6 – Math 435 – Summer

1. Solve $u_{xx} + u_{yy} = 0$ on the rectangle $0 < x < 2$, $0 < y < 3$, with boundary conditions

$$u_x(0, y) = 0$$
$$u_x(2, y) = 0$$
$$u(x, 0) = 0$$
$$u(x, 3) = 3x$$

2. Consider Laplace's equation on the disk $x^2 + y^2 < 9$ with boundary condition $u(x, y) = \frac{x^2}{3}$ when $x^2 + y^2 = 9$.

   (a) Without solving the equation, give the value of u at the center of the disk.

   (b) Solve the equation.

3. Use separation of variables to derive the solution $u(r, \theta)$ to Laplace's equation on the annulus $1 < r < 3$ with boundary conditions:

$$u(1, \theta) = \sin^2(\theta)$$
$$u(3, \theta) = 0$$

4. Consider the steady-state temperature distribution inside a spherical ball ($r < 4$), whose outer boundary sphere is held at a constant temperature of 10 degrees. According to the maximum principle for Laplace's equation in 3D, what can you conclude about $u(x, y, z)$ inside the ball? What does the minimum principle tell you?

5. Problem 5 from the exercises for section 7.1 in Strauss.

6. Show that the Green's function is unique for a given domain. (hint: take the difference of two of them and use the proof that solutions to Laplace's equation with Dirichlet b.c.'s are unique)

7. The Neumann function $N(x, y)$ for a domain D is defined exactly like the Green's function in Section 7.3 except that (ii) is replaced by the Neumann boundary condition

$$\frac{\partial N}{\partial n} = 0$$

   for $x \in \partial D$. In this case, we get the analogous statement to Theorem 7.3.1: If $N(x, x_0)$ is the Neumann function, then the solution of the Dirichlet problem is given by the formula

$$u(x_0) = -\int\int_{\partial D} N(x, x_0) \frac{\partial u}{\partial n}\, dS\ .$$

   Show this is true.

8. Use Green's functions to solve

$$\Delta u = 0$$

   inside the ball of radius $r = 3$, if we assume that $u = \frac{x+y}{3}$ on the boundary of the ball (the sphere of radius 3). Notice that the Green's function expression on a sphere that we talked about in class cannot be used to find the value of u at the origin (why?). How **could** we find the value of u at the origin? (note: you don't actually have to find the value, just describe how to)