

1. The solution is $u(x, t) = \frac{1}{2} (e^{x+ct} + e^{x-ct}) + \frac{1}{2c} \int_{x-ct}^{x+ct} \sin(s) dx$ or

$$u(x, t) = \frac{1}{2} (e^{x+ct} + e^{x-ct}) + \frac{1}{2c} (-\cos(x + ct) + \cos(x - ct))$$

2. To solve the PDE $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, you can first factor the differential operator to see

$$\left(\frac{\partial}{\partial x} - 4 \frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right) u = 0 .$$

Now you can either change variables according to the two different directions of characteristics that arise from the two factors of the operator, or you can directly find a change of variable so that $\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x} - 4 \frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \eta} = \frac{\partial}{\partial x} + \frac{\partial}{\partial t}$. To be consistent with the way we solved the wave equation in class, I will take the perspective of the characteristics.

Thinking of the first factor, it looks like a transport operator $\frac{\partial}{\partial x} - 4 \frac{\partial}{\partial t}$, and the characteristics for the associated transport equation would be given by the solution of the ode

$$\frac{dx}{dt} = \frac{1}{-4} .$$

The solution gives the characteristics $x = -\frac{t}{4} + C$, or $x + \frac{t}{4} = C$. Thus we could set

$$\xi = x + \frac{t}{4} .$$

Similarly, the other factor gives the transport operator $\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$, and it's characteristics are the solutions of

$$\frac{dx}{dt} = 1 .$$

Thus, the characteristics are $x - t = C$, and we can take

$$\eta = x - t$$

.

Thus,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} .$$

And we have

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial t} = \frac{1}{4} \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} .$$

Subbing into our factors for our PDE operator we have

$$\frac{\partial}{\partial x} - 4 \frac{\partial}{\partial t} = 5 \frac{\partial}{\partial \eta}$$

and

$$\frac{\partial}{\partial x} + \frac{\partial}{\partial t} = \frac{5}{4} \frac{\partial}{\partial \xi} .$$

Thus the PDE transforms to

$$\frac{25}{4} u_{\xi \eta} = 0 .$$

We can divide both sides by $\frac{25}{4}$ and then integrate with respect to η to get

$$u_\xi = f(\xi)$$

for some function f . Following with integration in ξ , we have

$$u = F(\xi) + G(\eta) ,$$

where $F(\xi) = \int f(\xi) d\xi$. Finally, changing variables back to x and t , we get the general solution to the original PDE is

$$u = F\left(x + \frac{t}{4}\right) + G(x - t) .$$

Now if we apply the initial conditions, we have:

$$u(x, 0) = F(x) + G(x) = \phi(x)$$

and

$$u_t(x, 0) = \frac{1}{4}F'(x) - G'(x) = \psi(x) .$$

Solving for F' gives

$$F'(x) = \frac{4}{5}(\phi'(x) + \psi(x))$$

and so

$$F(x) = \frac{4}{5}\phi(x) + \frac{4}{5}\int_0^x \psi(s) ds .$$

Now solving for G' gives

$$G'(x) = \frac{4}{5}\left(\frac{1}{4}\phi'(x) - \psi(x)\right)$$

or

$$G(x) = \frac{1}{5}\phi(x) - \frac{4}{5}\int_0^x \psi(s) ds .$$

Finally, we can obtain the unique solution to the initial value problem

$$u(x, t) = \left(\frac{4}{5}\phi\left(x + \frac{t}{4}\right) + \frac{1}{5}\phi(x - t)\right) + \frac{4}{5}\int_{x-t}^{x+\frac{t}{4}} \psi(s) ds .$$

3. The “damped wave equation” is given by

$$u_{tt} - c^2u_{xx} + ru_t = 0 .$$

Follow the basic outline of the proof of conservation of energy for the wave equation to show that for the damped wave equation, the total energy decreases over time if $r > 0$.

Solution

$$\begin{aligned} \frac{\partial}{\partial t} \int \frac{1}{2}\rho(u_t)^2 dx &= \int \rho u_t u_{tt} dx \\ &= \int \rho u_t (c^2 u_{xx} - r u_t) dx = -r\rho \int (u_t)^2 dx - [u_t u_x]_{-\infty}^{\infty} - \rho c^2 \int u_{tx} u_x dx \\ &= -r\rho \int (u_t)^2 dx - \frac{T}{2} \int \frac{\partial}{\partial t} (u_x)^2 dx \\ &= -r\rho \int (u_t)^2 dx - \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \frac{1}{2}T(u_x)^2 dx \end{aligned}$$

Thus, we can see that

$$\frac{\partial KE}{\partial t} = -r\rho \int (u_t)^2 dx - \frac{\partial PE}{\partial t}$$

and we can rearrange this to see that

$$\frac{\partial(KE + PE)}{\partial t} = -r\rho \int (u_t)^2 dx \leq 0 .$$

Thus, the total energy of the wave is decreasing over time for the damped wave equation.

4. Part (a): Since the minimum value of $u(x, 0) = 4x(1-x)$ on the interval $[0, 1]$ is 0 and since the maximum value of $u(x, 0)$ is 1 (which occurs at $x = 1/2$), and also since $u(1, t) = 0 = u(0, t)$, then by the strong maximum and minimum principles for the diffusion equation we have that $0 < u(x, t) < 1$ for any $0 < x < 1$ and $t > 0$.

Part(c):

$$\frac{\partial}{\partial t} \int_0^1 u^2 dx = \int_0^1 2uu_t dx = \int_0^1 u(ku_{xx}) dx = ku u_x|_0^1 - k \int_0^1 (u_x)^2 dx = -k \int_0^1 (u_x)^2 dx < 0$$

Thus, since the time derivative of the integral is negative, it is a strictly decreasing function over time. (Note we do not get that the final integral above can equal zero because we cannot have $u_x(x, t) = 0$ for all x and t , as this would imply that u is constant in x or $u(x, t) = f(t)$. This in turn would imply, since $u_t = u_{xx} = 0$ that $u(x, t) = C$. The only way $u = C$ is possible is if $u = 0$ to satisfy the boundary conditions. But this doesn't agree with our initial data.)

5. Part (a): Verifying the solution: $u_x = -2t - 2x$, $u_{xx} = -2$ and $u_t = -2x$, thus $u_t = xu_{xx}$. The critical points of $u(x, t)$ occur when both $u_x = 0$ and $u_t = 0$, so that $t = -x$ and $x = 0$, which implies that $t = x = 0$. At $(0, 0)$, the concavity in the x -direction is $u_{xx} = -2$ and in the t -direction is $u_{tt} = 0$, so that the only kind of extremum the point can be is a local maximum if it's an extremum at all. We get no extrema on the interior of the region. Checking the boundary pieces, we see that on $x = -2$, $u(-2, t) = 4t - 4$ which has its maximum value at $t = 1$ and gives a max value of $u(-2, 1) = 0$. On $x = 2$, we have $u(2, t) = -4t - 4$ which has its maximum value at $t = 0$, giving a max value of -4 . On $t = 0$, $u(x, 0) = -x^2$ which has its max value at $x = 0$ and the max value is 0. Finally, on $t = 1$, we see $u(x, 1) = -2x - x^2$ whose maximum occurs at $x = -1$ giving a maximum value of $u(-1, 1) = 1$. Comparing the values of u over the boundary and at the critical point we see that the absolute maximum value of u is 1 and it occurs at the point $(x, t) = (-1, 1)$. Note that this point is NOT on $x = 2$, $x = -2$ or $t = 0$, so that the statement of the maximum principle is not true for this equation.

Part(b): Again letting $v(x, t) = u(x, t) + \epsilon x^2$ and differentiating v with the same operator as the PDE, we have:

$$v_t - xv_{xx} = u_t - xv_{xx} - 2k\epsilon = -2k\epsilon$$

Let M be the max value of u on the boundary edges $x = -2$, $x = 2$ and $t = 0$. Then on those same edges, $v(x, t) \leq M + \epsilon l^2$. If v has an interior maximum at (x_0, t_0) , then $v_t(x_0, t_0) = 0$ and $v_{xx}(x_0, t_0) \leq 0$. Thus, $v_t - xv_{xx} = -xv_{xx}$, but because x could be positive or negative, we cannot say anything about the sign of this expression! Hence we cannot arrive at the same contradiction we saw in the regular diffusion equation case.

6. Let $w = u - v$. Then $w_t - kw_{xx} = (u - v)_t - k(u - v)_{xx} = 0$ and $w(0, t) \leq 0$, $w(l, t) \leq 0$ and $w(x, 0) \leq 0$. By the (strong) maximum principle applied to w (we can apply it to w because it's a solution of the heat equation), we see that $w(x, t) < 0$ for all x in $(0, l)$ and $t > 0$. Hence, $u - v < 0$ for all x in $(0, l)$ and $t > 0$, or $u < v$ for all x in $(0, l)$ and $t > 0$.

7.
$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y) dy = \frac{3}{\sqrt{4\pi kt}} \int_{-\infty}^0 e^{-(x-y)^2/4kt} dy + \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} e^{-(x-y)^2/4kt} dy$$

Again, letting $p = \frac{x-y}{\sqrt{4kt}}$, we have $dp = -\frac{1}{\sqrt{4kt}} dy$ and subbing in, we have

$$u(x, t) = -\frac{3}{\sqrt{\pi}} \int_{\infty}^{x/\sqrt{4kt}} e^{-p^2} dp - \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4kt}}^{-\infty} e^{-p^2} dp .$$

Rearranging in order to use the error function, this becomes

$$u(x, t) = -\frac{3}{\sqrt{\pi}} \int_{\infty}^0 e^{-p^2} dp - \frac{3}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-p^2} dp - \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4kt}}^0 e^{-p^2} dp - \frac{1}{\sqrt{\pi}} \int_0^{-\infty} e^{-p^2} dp$$

or

$$u(x, t) = \frac{3}{2} \operatorname{erf}(\infty) - \frac{3}{2} \operatorname{erf}(x/\sqrt{4kt}) + \frac{1}{2} \operatorname{erf}(x/\sqrt{4kt}) - \frac{1}{2} \operatorname{erf}(-\infty) .$$

Since $\operatorname{erf}(\infty) = 1$, and by symmetry $\operatorname{erf}(-\infty) = -1$, we have

$$u(x, t) = 2 - \operatorname{erf}(x/\sqrt{4kt}) .$$

Note that as $t \rightarrow \infty$, $u(x, t)$ tends to the steady state $u(x, t) = 2$.

8. If $u(x, 0) = e^{3x}$, then

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} e^{3y} dy$$

We can begin to rearrange our terms so that we have a power in the form of a perfect square within the integrand:

$$u(x, t) = \frac{e^{-x^2/4kt}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(-2xy+y^2-12kty)/4kt} dy$$

completing the square gives

$$u(x, t) = \frac{e^{(-x^2+(-x-6kt)^2)/4kt}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-((-2x-12kt)y+y^2+(-x-6kt)^2)/4kt} dy$$

or

$$u(x, t) = \frac{e^{9kt+3x}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(y-x-6kt)^2/4kt} dy$$

Now we can make the substitution $p = \frac{y-x-6kt}{\sqrt{4kt}}$, we have $dp = \frac{1}{\sqrt{4kt}} dy$, and

$$u(x, t) = \frac{e^{9kt+3x}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} dp$$

or

$$u(x, t) = e^{9kt+3x} .$$

9. We make the change of variables $u(x, t) = e^{-bt}v(x, t)$, so the $u_t = -be^{-bt}v + e^{-bt}v_t$ and $u_{xx} = e^{-bt}v_{xx}$. Subbing into the PDE, we have

$$-be^{-bt}v + e^{-bt}v_t - ke^{-bt}v_{xx} + be^{-bt}v = e^{-bt}(v_t - kv_{xx}) = 0$$

Since e^{-bt} is nowhere zero, we can divide by it and get

$$v_t - kv_{xx} = 0 .$$

Thus, since $u(x, 0) = e^0v(x, 0) = v(x, 0) = \phi(x)$

$$v(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y) dy$$

and

$$u(x, t) = \frac{e^{-bt}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y) dy$$

is the solution to the diffusion equation with constant dissipation.

10. Now we make the change of variable $y = x - Vt$ so that we are tracking space by a moving frame of reference (our position y depends on time) and keep time measured in the same way so that our new time variable \hat{t} is $\hat{t} = t$. Then $u_x = u_y y_x + u_{\hat{t}} \hat{t}_x = u_y$ and $u_t = u_y y_t + u_{\hat{t}} \hat{t}_t = -V u_y + u_{\hat{t}}$. Subbing into the PDE, we have

$$u_{\hat{t}} - k u_{yy} = 0$$

Thus, since we also know that when $t = 0$, $y = x$ and $\hat{t} = 0$, then $u(y, 0) = \phi(y)$, and

$$u(y, \hat{t}) = \frac{1}{\sqrt{4\pi k\hat{t}}} \int_{-\infty}^{\infty} e^{-(y-s)^2/4k\hat{t}} \phi(s) ds$$

or the solution of the diffusion equation with transport over the whole real line is:

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-Vt-s)^2/4kt} \phi(s) ds .$$