Homework Set # 2 Solutions – Math 435 – Summer 2013

1. Suppose that we have a uniform thin tube (approximable by one space dimension) of liquid with some particles which are suspended in the liquid. If the liquid is flowing through the pipe at a constant rate c (m/s) and if we also take into account that the particles diffuse within the solution, derive the PDE for the concentration of the particles $u(x,t)$.

   Solution:

   Let's follow a "slice of fluid" between x_0 and x_1 at $t = t_0$ and look at it again at $t = t_0 + h$. The only way we can lose mass in this slice if it some diffuses out either end, otherwise, the mass is transported along the tube in the slice without being lost. Thus we can say that since, if M represents the total mass of particles between x_0 and x_1,

   $$M = \int_{x_0}^{x_1} u(x,t_0)\,dx$$

   then at the later time $t_0 + h$, the mass in the same transported slice would be:

   $$M = \int_{x_0+ch}^{x_1+ch} u(x,t_0+h)\,dx \ .$$

   Any difference is these two masses is due to diffusion, so that the net change in the mass between times t_0 and $t_0 + h$ is given by

   $$\int_{x_0+ch}^{x_1+ch} u(x,t_0+h)\,dx - \int_{x_0}^{x_1} u(x,t_0)\,dx = \int_{t_0}^{t_0+h} ku_x(x_1,t) - ku_x(x_0,t)\,dt$$

   because the rate of diffusion is always proportional to the spacial gradient of the concentration as we discussed in setting up the diffusion equation, and thus the instantaneous rate of change of the mass due to diffusion in the slice $[x_0, x_1]$ at time t is

   $$\frac{dM}{dt} = ku_x(x_1,t) - ku_x(x_0,t) \ .$$

   If we differentiate our equality with respect to x_1, we get

   $$u(x_1 + ch, t_0 + h) - u(x_1, t_0) = \int_{t_0}^{t_0+h} ku_{xx}(x_1,t)\,dt \ .$$

   Differentiating with respect to h now gives

   $$cu_x(x_1 + ch, t_0 + h) + u_t(x_1 + ch, t_0 + h) = ku_{xx}(x_1, t_0 + h) \ .$$

   Since h is arbitrary, this is true for any value of h, and in particular this equality holds when $h = 0$, so

   $$cu_x(x_1,t_0) + u_t(x_1,t_0) = ku_{xx}(x_1,t_0) \ .$$

   Again, we notice that x_1 and t_0 were arbitrary, so

   $$cu_x + u_t = ku_{xx}$$

   for all x and t. This is the equation that models transport and diffusion combined.

2. Suppose now that we have a still fluid in a tube, and again have particles suspended in that liquid. The particles move by diffusion AND sediment out of the solution at a fixed percentage rate v (in units 1/s - v is the fraction of particles that fall out of solution per second). Derive the PDE modeling the concentration of the particles $u(x,t)$.

   Solution:

   If we look at a small segment of fluid then between x_0 and x_1, we can try to describe the rate of change of the mass of particulate in that small segment.

   The total mass in the segment at time t is:

$$M = \int_{x_0}^{x_1} u(x,t)dx$$

   so the rate of change of the mass in the segment is

$$\frac{dM}{dt} = \int_{x_0}^{x_1} u_t(x,t)dx \ .$$

   On the other hand the particles are moving solely due to diffusion and by sedimentation out of solution. Thus, the rate of change of the mass in the segment can also be described by:

$$\frac{dM}{dt} = \text{rate of change due to sedimentation} + \text{rate of change due to diffusion}$$

$$= -v\int_{x_0}^{x_1} u(x,t)\,dx + \frac{dM_{diff}}{dt} = -v\int_{x_0}^{x_1} u(x,t)\,dx + ku_x(x_1,t) - ku_x(x_0,t)$$

   This holds because if we are only considering the effect of the sedimentation, we have a percentage of particles falling out of solution in any given second, given by our rate of v (measured in 1/sec) and we want $\frac{dM}{dt}$ which is measured in units of mass/time. Thus we need to multiply our percent rate v by the mass of particles in our small interval $\int_{x_0}^{x_1} u(x,t)\,dx$.

   Finally, putting together our two estimations of $\frac{dM}{dt}$, we see

$$\int_{x_0}^{x_1} u_t(x,t)dx = -v\int_{x_0}^{x_1} u(x,t)\,dx + ku_x(x_1,t) - ku_x(x_0,t)$$

   which we can differentiate with respect to x_1 to get

$$u_t(x_1,t) = -vu(x_1,t) + ku_{xx}(x_1,t) \ .$$

   Since x_1 is arbitrary, $u_t = -vu + ku_{xx}$ over the whole domain.

3. Suppose that a uniform rod (approximable as one-dimensional) has a uniform heat source, so that the basic equation describing heat flow within the rod is

$$u_t = \alpha^2 u_{xx} + 1$$

   for $0 \le x \le 1$. Suppose we fix the boundaries' temperatures so that at $x = 0$ the rod is held at temperature 0 and at $x = 1$ the rod is held at temperature 1.

   (a) Formulate the boundary conditions for the given problem.

(b) Write the boundary value problem (meaning the PDE and the boundary conditions) that describes the steady-state temperature of the rod.

(c) Use ODE techniques to solve the steady-state problem, if possible.

Solutions:

(a) $u(0,t) = 0$ and $u(1,t) = 1$

(b) Steady-state implies that the system is not changing over time, so we let $u_t = 0$. This gives us the BVP

$$u_{xx} = -\frac{1}{\alpha^2}$$

with $u(0) = 0$ and $u(1) = 1$. Note that u no longer depends on time because it is a steady state solution!

(c) $u_{xx} = \frac{-1}{\alpha^2}$ implies that $u_x = \frac{-1}{\alpha^2}x + c$ after integrating both sides with respect to x. If we integrate once more we get $u = \frac{-1}{2\alpha^2}x^2 + cx + b$. Now, implementing the boundary conditions, we get $u(0) = b = 0$ and $u(1) = c - \frac{1}{2\alpha^2} = 1$. Thus our steady state solution is $u(x) = \frac{-1}{2\alpha^2}x^2 + \left(1 + \frac{1}{2\alpha^2}\right)x$.

4. (a) What is your interpretation of the initial-boundary-value problem:

$$\begin{aligned} u_t &= \alpha^2 u_{xx} \qquad \text{for } 0 \le x \le 1 \text{ ,} \qquad 0 < t < \infty \\ u(0,t) &= 0 \\ u_x(1,t) &= 1 \qquad \text{for} 0 < t < \infty \\ u(x,0) &= sin(\pi x) \qquad \text{for } 0 \le x \le 1 \end{aligned}$$

(b) Can the solution come to a steady state? [hint: try to find steady-state solutions]

(c) Answer (a) and (b) again, but with the boundary conditions

$$\begin{aligned} u_x(0,t) &= 0 \\ u_x(1,t) &= 0 \qquad \text{for} 0 < t < \infty \end{aligned}$$

(a) This system models 1D diffusion (or heat flow) over an interval of length 1 such that the particle density (or temp) is held at zero at the left end and there is a constant positive flux of particles at the right end into the domain (or there is constant heat flow into the rod at the right end). The inital particle density (or heat distribution) is given by $\sin(\pi x)$ over the domain.

(b) If we are at steady state, we would have $u_t = 0$ and so again $u_{xx} = 0$, with boundary conditions $u(0) = 0$ and $u_x(1) = 1$. Solving as before, we have $u = cx + b$. $u(0) = b = 0$ and $u_x(1) = c = 1$. So we again get $u(x) = x$ as a steady state solution. Thus the solution CAN come to steady state.

(c) If we now change the boundary conditions, the physical interpretation becomes that there is no flux of particles (or no heat flow) through either end of the domain. For the steady state situation, we get $u = cx + b$, but now $u_x(0) = c$ and $u_x(1) = c$ which both imply that $c = 0$ must be true. This gives $u = b$ as the steady state solutions. So, we actually have infinitely many possible steady state solutions in this case - the particle density is uniform throughout the domain regardless of the value of the density (or the temperature is constant throughout the domain regardless of the value of the temperature).

5. (a) What is your interpretation of the initial-boundary-value problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \qquad \text{for } 0 \le x \le 1 \ , \qquad 0 < t < \infty \\ u(0,t) &= 0 \\ u(1,t) &= \sin(t) \qquad \text{for} 0 < t < \infty \\ u(x,0) &= 0 \\ u_t(x,0) &= 0 \qquad \text{for } 0 \le x \le 1 \end{aligned}$$

(b) Can the solution come to a steady state?

Solutions

(a) This models the wave equation on a 1D string of length 1, where the left end is fixed at height zero and the right end is oscillated up and down with the height given by $\sin(t)$ at time t. The inital position is at equillibrium (height zero everywhere) and the initial velocity is zero.

(b) Looking for steady state solutions, we assume $u_t = 0$, and so $u_{tt} = 0$. This gives us $u_{xx} = 0$. Our boundary conditions remain $u(0) = 0$ and $u(1) = \sin(t)$. As we solve in the same way as for the previous problems, we obtain $u = cx + b$. But in trying to implement the boundary values, we see $u(0) = b = 0$, and $u(1) = c = \sin(t)$ which does not make sense since c is constant and $\sin(t)$ is clearly not. Thus this system has NO steady state solutions, and the physical system never comes to a steady state. [Note that this is physically reasonable since the right end is oscillated according to the sine function FOR ALL TIME. This implies motion of the entire string for all time, which does not allow for the position of the string to remain fixed over time]

6. Section 1.5 Strauss, problem 5. Consider the equation

$$u_x + y u_y = 0$$

with boundary conditions $u(x,0) = \phi(x)$.

(a) For $\phi(x) = x$ show no solution exists.

(b) for $\phi(x) = 1$ show infinitely many solutions exist.

Solution

Solving the transport equation requires that we find the characteristics by solving the ODE

$$\frac{dy}{dx} = y \ .$$

By separation, we can see that $ln|y| = x + c$, or $y = Ae^x$. Thus the general solution to the equation is $u(x,y) = f(ye^{-x})$. Applying the boundary condition $u(x,0) = x$, we get

$$u(x,0) = f(0) = x \ .$$

But $f(0)$ is necessarily a constant, and so this equality cannot hold for all x. There is no solution to this BVP.

On the other hand for part (b), we would have

$$u(x,0) = f(0) = 1 \ .$$

Since there are infinitely many functions f such that $f(0) = 1$ (like $f(x) = x + 1$, $f(x) = cos(x)$, $f(x) = e^x$, etc), we have infinitely man choices for our function f and so infintely many solutions to the BVP (like $u(x, y) = ye^{-x} + 1$, $u(x, y) = cos(ye^{-x})$, $u(x, y) = e^{ye^{-x}}$, etc..)

7. Section 1.5, problem 6. Solve the equation $u_x + 2xy^2u_y = 0$ with $u(x, 0) = phi(x)$.

   Solution

   Again we solve the transport equation by solving this ODE for the characteristics:

$$\frac{dy}{dx} = 2xy^2$$

   By separation, we see that

$$-\frac{1}{y} = x^2 + c$$

   and the general solution to the transport equation is

$$u(x, y) = f(x^2 + \frac{1}{y}) \ .$$

   If we apply the boundary condition, we have

$$u(x, 0) = f(x^2 + 1/0)$$

   Oh My! Thus, this is not a valid boundary condition for this PDE and no solution exists.

8. What are the types of the following equations (elliptic, parabolic, or hyperbolic)?

   (a) $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0$

   (b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$

   (c) $u_{xx} - 4u_{xy} + 4u_{yy} = 0$

   (d) $u_{xx} - 4u_{xy} - 4u_{yy} = 0$

   Solutions:

   (a) $a_{11} = 1$, $a_{22} = 1$, $a_{12} = -2$ (since $u_{xy} = u_{yx}$) so $a_{12}^2 = 4 > 1 = a_{11}a_{22}$ and the equation is hyperbolic.
   (b) $a_{11} = 9$, $a_{22} = 1$, $a_{12} = 3$ so $a_{12}^2 = 9 = a_{11}a_{22}$ and the equation is parabolic.
   (c) $a_{11} = 1$, $a_{22} = 4$, $a_{12} = -2$ so $a_{12}^2 = 4 = a_{11}a_{22}$ and the equation is parabolic.
   (d) $a_{11} = 1$, $a_{22} = -4$, $a_{12} = -2$ so $a_{12}^2 = 4 > a_{11}a_{22}$ and the equation is hyperbolic.

9. Section 1.6, Problem 2. Find the regions in the xy-plane where the equation

$$(1 + x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

   is elliptic, hyperbolic, or parabolic.

   Solution

   $a_{11} = 1 + x$, $a_{22} = -y^2$ and $a_{12} = xy$, so it is

(a) elliptic where $x^2y^2 < -y^2(1+x)$. Note, for this inequality to hold, $y \neq 0$, so that we can divide both sides by y^2 and get $x^2 < -(1+x)$. This can never happen! So this PDE is nowhere elliptic.

(b) parabolic where $x^2y^2 = -y^2(1+x)$. This can occur either for $y = 0$ or $x^2 = -(1+x)$. Since the latter is impossible, (note that $x^2 + x + 1 = 0$ has no real solutions) the PDE is only parabolic on the line $y = 0$ (the x-axis).

(c) hyperbolic where $x^2y^2 > -y^2(1+x)$. again, this can only hold if $y \neq 0$, so we actually need $x^2 > -(1+x)$, or $x^2+x+1 > 0$. This is true for all x! Hence the PDE is hyperbolic everywhere except the x-axis.

5. Use the rotational change of variables:

$$x = \xi \cos\theta - \eta \sin\theta$$
$$y = \xi \sin\theta + \eta \cos\theta$$

or equivalently:

$$\xi = x\cos\theta + y\sin\theta$$
$$\eta = -x\sin\theta + y\cos\theta$$

for some angle of rotation θ, to show that any equation of the form $au_{xx} + au_{yy} + bu = 0$ is invariant under rotation (the form of the equation doesn't change under the change of variables!).

Solution By the given change of variables, we have that

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial \xi} - \sin\theta\frac{\partial}{\partial \eta}$$

and

$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial \xi} + \cos\theta\frac{\partial}{\partial \eta}\ .$$

Thus,

$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial \xi} - \sin\theta\frac{\partial}{\partial \eta}\right)\left(\cos\theta\frac{\partial u}{\partial \xi} - \sin\theta\frac{\partial u}{\partial \eta}\right)$$

and

$$u_{xx} = \cos^2\theta u_{\xi\xi} - 2\cos\theta\sin\theta u_{\xi\eta} + \sin^2\theta u_{\eta\eta}\ .$$

Similarly

$$u_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\sin\theta\frac{\partial}{\partial \xi} + \cos\theta\frac{\partial}{\partial \eta}\right)\left(\sin\theta\frac{\partial u}{\partial \xi} + \cos\theta\frac{\partial u}{\partial \eta}\right)$$

and

$$u_{yy} = \sin^2\theta u_{\xi\xi} + 2\cos\theta\sin\theta u_{\xi\eta} + \cos^2\theta u_{\eta\eta}\ .$$

So subbing into our PDE, $au_{xx} + au_{yy} + bu = 0$ gives

$$a(\cos^2\theta u_{\xi\xi} - 2\cos\theta\sin\theta u_{\xi\eta} + \sin^2\theta u_{\eta\eta}) + a(\sin^2\theta u_{\xi\xi} + 2\cos\theta\sin\theta u_{\xi\eta} + \cos^2\theta u_{\eta\eta}) + bu = 0$$

and simplified we get

$$au_{\xi\xi} + au_{\eta\eta} + bu = 0\ .$$

Thus, this PDE is called "rotationally invariant".