Homework Set # 1 SOLUTIONS – Math 435 – Spring 2013 Due date: 1/14/2013

1. Determine whether or not the following functions are solutions to the given PDE.

   (a) $u_x - 3u_y = 0$, $u(x, y) = \cos(y + 3x)$

   (b) $u_x - 3u_y = 0$, $u(x, y) = 9x^2 + 6xy + y^2$

   (c) $u_{xx} + u_{yy} = 0$, $u(x, y) = x^2 + y^2$

   (d) $u_{xx} - 2u_t = 0$, $u(t, x) = e^t[2e^{\sqrt{2}x} + 3e^{-\sqrt{2}x}]$

   Solutions:

   (a) $u_x = -3\sin(y + 3x)$ and $u_y = -\sin(y + 3x)$, so $u_x - 3u_y = 0$ is true. YES.

   (b) $u_x = 18x + 6y$ and $u_y = 6x + 2y$, so that $u_x - 3u_y = 0$ holds. YES.

   (c) $u_x = 2x$, so $u_{xx} = 2$ and $u_y = 2y$ so $u_{yy} = 2$. Thus $u_{xx} + u_{yy} = 4 \neq 0$. NO.

   (d) $u_x = e^t[2\sqrt{2}e^{\sqrt{2}x} - 3\sqrt{2}e^{-\sqrt{2}x}]$, so that $u_{xx} = e^t[4e^{\sqrt{2}x} + 6e^{-\sqrt{2}x}]$. Also $u_t = e^t[2e^{\sqrt{2}x} + 3e^{-\sqrt{2}x}]$, so $-2u_t + u_{xx} = 0$ holds. YES.

2. For the following PDE, determine whether or not they are linear (prove it!) and give their order.

   (a) $u_x + xu_y = 0$

   (b) $u_x + uu_y = 0$

   (c) $u_x + u_y + 1 = 0$

   (d) $u_x + (u_y)^2 = 0$

   Solutions:

   (a) This PDE is order 1, because there are only first order derivatives, and is homogeneous. Expressed in operator notation in standard form, we have $L(u) = 0$ where $L(u) = u_x + xu_y$. Since

   $$L(u+v) = (u+v)_x + x(u+v)_y = u_x + v_x + x(u_y + v_y) = (u_x + xu_y) + (v_x + xv_y) = L(u) + L(v)$$

   and since

   $$L(cu) = (cu)_x + x(cu)_y = cu_x + xcu_y = c(u_x + xu_y) = cL(u)$$

   we have that the PDE is linear.

   (b) This PDE is order 1. Expressed in operator notation in standard form, we have $L(u) = 0$ where $L(u) = u_x + uu_y$, so it is also homogenous. Since

   $$\begin{aligned} L(u+v) &= (u+v)_x + (u+v)(u+v)_y = u_x + v_x + (u+v)(u_y + v_y) \\ &= (u_x + uu_y) + vu_y + (v_x + vv_y) + uv_y \neq L(u) + L(v) \end{aligned}$$

   so that the PDE is not linear.

(c) This PDE is order 1, because there are only first order derivatives. Expressed in operator notation in standard form, we have $L(u) = -1$ where $L(u) = u_x + u_y$, thus it is inhomogenous. Since

$$L(u+v) = (u+v)_x + (u+v)_y = u_x + v_x + u_y + v_y = (u_x + u_y) + (v_x + v_y) = L(u) + L(v)$$

and since

$$L(cu) = (cu)_x + (cu)_y = cu_x + cu_y = c(u_x + u_y) = cL(u)$$

we have that the PDE is linear (but inhomogeneous).

(d) This PDE is order 1, because there are only first order derivatives. Expressed in operator notation in standard form, we have $L(u) = 0$ where $L(u) = u_x + (u_y)^2$, and is homogeneous. Since

$$L(u+v) = (u+v)_x + ((u+v)_y)^2 = u_x + v_x + (u_y + v_y)^2 = (u_x + (u_y)^2) + (v_x + (v_y)^2) + 2u_y v_y \neq L(u) + L(v)$$

so that the PDE is not linear.

3. Find the general solution to the PDE $u_{yy} - u = 0$.

Solution

We can solve this PDE as an ODE since the derivatives occur only in the variable y. We can try solutions of the form $u = e^{ry}$, and get that $r^2 - 1 = 0$ must be true. Thus $r = \pm 1$, and the general solution to the PDE is $u(x, y) = c_1(x)e^y + c_2(x)e^{-y}$.

4. Suppose you have a linear homogeneous PDE $L(u) = 0$. Suppose that $u_1, u_2, \ldots, u_n$ are all solutions to the PDE. Show that any linear combination of those solutions is itself a solution to the PDE. Show that this is not true if the PDE is inhomogeneous. (Note: This is the superposition principle for PDE.)

Solution:

Since u_i satisfies $L(u_i) = 0$ for all $i \in \{1, \dot; n\}$, since a linear combination of the u_i is $v = c_1u_1 + c_2u_2 + \cdots + c_nu_n$ for some choice of constants c_i, and since the operator for our PDE is linear, we have

$$L(v) = L(c_1u_1 + \cdots + c_nu_n) = c_1L(u_1) + c_2L(u_2) \ldots c_nL(u_n) = c_1 * 0 + \cdots + c_n * 0 = 0\ .$$

Thus $L(v) = 0$ also and the linear combination v is a solution of the PDE. Since v is an arbitrary linear combination of the u_i's, ANY linear combination of the u_i's is a solution to the PDE.

Now, if the PDE is inhomogeneous, so that $L(u) = g$ for some function g not identically zero, we will instead have

$$L(v) = L(c_1u_1 + \cdots + c_nu_n) = c_1L(u_1) + c_2L(u_2) \ldots c_nL(u_n) = c_1 * g + \cdots + c_n * g \neq g\ .$$

So, $L(v) \neq g$ and the linear combination is no longer solution to the PDE.

5. Verify that $u(x, y) = f(x)g(y)$ is a solution of the PDE $uu_{xy} = u_xu_y$ for all pairs of (differentiable) functions f and g of one variable.(Strauss Problem 11, Section 1.1)

Solution:

$$u_x(x,y) = f'(x)g(y)$$
$$u_y(x,y) = f(x)g'(y)$$

Thus, $u_{xy} = f'(x)g'(y)$. Plugging into $uu_{xy} = u_x u_y$, we get:

$$f(x)g(y)f'(x)g'(y) = f'(x)g(y)f(x)g'(y)$$

which is a true statement! So the given u is in fact a solution of the PDE.

6. The following parts refer to the PDE $2u_t + 3u_x = 0$

   (a) Find the general solution to the PDE. Check that it does in fact solve the PDE.

   (b) If you add the condition that $u(x,0) = sin(x)$, what is $u(x,t)$?

   (c) Sketch the solution $u(x,t)$ for $t = 0, 1, 2$.

   (d) What is the speed of propagation for the transport equation?

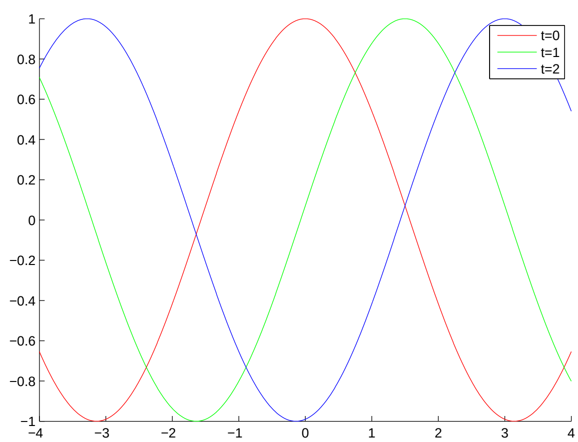
   Solution:

   (a) the general solution is $u(x,t) = f(2x - 3t)$ or $u(x,t) = f(t - \frac{2}{3}x)$ or $u(x,t) = f(x - \frac{3}{2}t)$.

   (b) If $u(x,0) = \cos(x)$, then since $u(x,0) = f(2x)$ using the first general solution from part (a), we have

   $$f(2x) = \cos(x)$$

   which implies that $f(x) = \cos(x/2)$. Thus, $u(x,t) = f(2x - 3t) = \cos(x - \frac{3}{2}t)$.



   (c)

   (d) note that the equation can be written as $< 2, 3 > \cdot < u_t, u_x >= 0$ so that the ODE for the characteristics is $\frac{dx}{dt} = \frac{3}{2}$, and thus the speed of propagation is $c = 3/2$. Also, note in the graphs above that for every unit of time, the solution is shifted to the right by 3/2 units in space. speed = change in position/change in time = (3/2)/1.

7. The following parts refer to the PDE $e^x u_x + u_y = 0$.

   (a) Find the characteristic curves for this PDE and sketch at least 4 of them in the same plane.

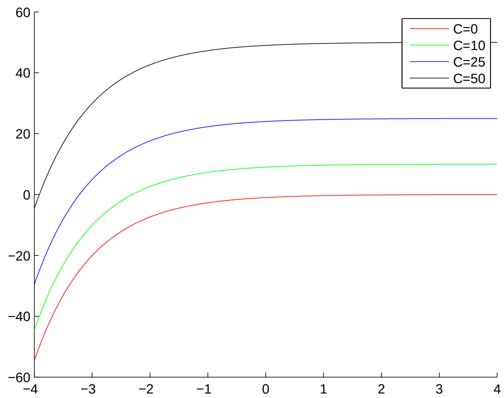
   (b) Find the general solution to the PDE.

(c) If $u(0, y) = (y - 2)^2$, find $u(x, y)$.

(d) Sketch $u(x, y)$ for $y = 0, 1, 2$. Note that we no longer have the same behavior for the solutions "over time" as we had in the constant coefficient case.

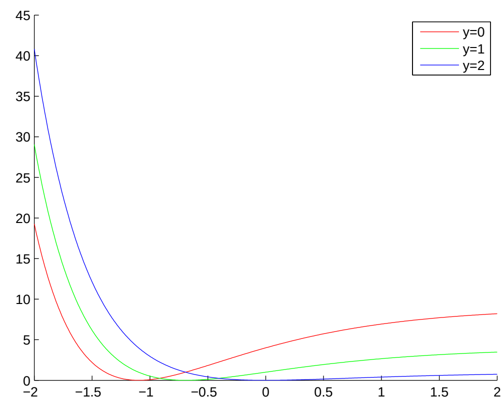
Solution:

(a) We can find the characteristic curves by rewriting the equation as $u_x + e^{-x} u_y = 0$, and setting $\frac{dy}{dx} = e^{-x}$. Solving this new ODE, we obtain $y = -e^{-x} + C$ which are the characteristic curves. Sketches of the characteristics are below for $C = 0, 10, 25, 50$.



(b) The general solution to the PDE is then $u(x, y) = f(y + e^{-x})$.

(c) By part (b), $u(0, y) = f(y+1)$ and by the fact that $u(0, y) = (y-2)^2$, if we let $w = y+1$ we can see that we must have $f(w) = (w-3)^2$. Thus, the solution is $u(x, y) = (y+e^{-x}-3)^2$.



(d)