Department of Mathematics
The University of Tennessee

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H. Finotti

Math 231:Introduction to Ordinary Differential Equations
Mini-Project: Rangeland Ecosystems

Concern for the quality of public lands has received much attention in recent years. The condition of rangelands administered by the Bureau of Land Management (BLM), and other Federal agencies, has been one of the issues in this concern. The BLM manages public lands that are, for the most part, arid and covered with low scrub and grasses. Private ranchers pay the BLM a fee to be allowed to graze cattle on these lands. The cattle, in eating the vegetation on the land, actually change the balance of species in a region. This exercise deals with one aspect of the problem of the effect of cattle on rangelands. In particular, there is competition in the wild between natural perennial grasses and exotic annual grasses, principally cheatgrass. The perennial grasses are preferable for the cattle, both in terms of nutritive value, and apparently in taste. Thus the cattle tend to eat perennial grasses first, and resort to eating cheatgrass only after most of the perennials are gone. If the area colonized by the perennials becomes too small relative to that colonized by the annuals, it is possible for the perennials to be completely eliminated through competition.

A model has been introduced to allow these levels to be discussed [1]. A part of this model is the topic of this example. Suppose we have an area in which perennial grasses and annual grasses are in competition. Let g denote the fraction of patches in that area which is containing perennial grasses, and let w (for weeds!) denote the fraction of patches in that area that are colonized by annuals. We consider that grasses and weeds may share a patch of ground, so that both w and g may vary only from zero to one, but they need not sum to one. The equations describing their dynamics are

$$g' = R_g g \left(1 - g - k_g w \frac{E+g}{0.3E+g}\right) \tag{1}$$

$$w' = R_w w \left(1 - w - k_w g \frac{0.3E+g}{E+g}\right) \tag{2}$$

where R_g and R_w represent the intrinsic growth rates of the grasses and weeds, respectively. The cattle stocking rate is E, and E increases linearly with the number of cattle on the plot.

We want to find threshold values of g and w dividing the case in which the grasses are able to compete successfully with the weeds, from the case in which the perennial grasses are so sparse that they can no longer compete, and die off in the area. We are interested in the phase plane only in the unit square. In this part of the plane, we have the potential in some cases for several equillibria!

My intention for a group of three is that each member of the group have primary responsibility for one of numbers 1,2, or 3. Numbers 4 and 5 should be done together as a group.

1. Let's begin by looking at what happens in the absence of the cattle (so let $E = 0$). Give the resulting system of differential equations, and describe the purpose of each of the terms in the equations. Do a full phase plane analysis for $R_g = 0.25$, $R_w = 0.4$, $k_g = 0.5$ and $k_w = 1.05$.

   Using MATLAB to approximate solutions to this system for at least three different initial conditions, chosen from different regions of the phase plane if possible. What are all the potential long term outcomes in the absence of grazing?

   As an experiment to understand the role of k_g and k_w, do the dynamics of the system change if we switch the values so that $k_g = 1.07$ and $k_w = 0.6$? Repeat the process above to find out. If so, explain physically why the changes occur in the dynamics based on the meaning of the change in the parameters.

2. Do a full phase plane analysis for $E = .25$, with all other parameters as in the first part of number one. For this and the following parts, please feel free to use graphing software to plot your nullclines and find their intersection points.

   Show that if the initial ratio of grasses and weeds corresponds to a point in the phase plane near the interior stable equillibrium, then the trajectories through that point will approach the stable equilibrium as time passes. On the other hand, if the point lies too far from the interior stable node, then the trajectory through it may approach some other stable node, such as the one at (g,w) = (0,1). The latter is an undesirable turn of events, since that equilibrium represents an all-weeds outcome. Verify these statements by using MATLAB to approximate solutions to this system for at least three different appropriate initial conditions. What are all the potential outcomes? Is there an outcome that seems more likely than the others (or outcomes)? Are there any clear statements you can make about what will determine whether or not the final outcome is a desirable one?

Repeat the above with $E = 0.5$ and all other parameters the same.

3. Leaving all other parameters the same as in the *second* part of problem 1, set $E = 0.25$. Repeat the work in Problem 2 with this new parameter.

   Now change the value of E to 0.1. Repeat the work in Problem 2 with this new parameter.

4. Summarize the results of problems one through three in physical terms. According to your results what would the most effective management strategy focus on in order to ensure the long term health of the grassland?

Reference: K. D. Cooper and R. Huffaker, The long term bioeconomic impacts of grazing on plant succession in a rangeland ecosystem, Ecological Modelling, 97 no. 1-2 (1997).