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Math 231:Introduction to Ordinary Differential Equations
Mini-Project: Vegetation-Erosion Dynamics

In this community and in many communities in this world, the effects of erosion on our rivers and ecosystems are often devastating. Recently, however, mathematicians and scientists have been inves- tigating the dynamic relationship between vegetation and erosion processes and the effects of human stresses on a particular watershed system. It is not surprising that increases in erosion will have an inverse relationship with the vegetation-cover density (VCD) in any given area. We may also find it easy to understand that human activities such as logging, hunting, building roads, and so on will affect vegetation development. Beyond human causes, however, certain ecological events such as wind storms, pests, drought, air pollution, and severe temperatures have been targeted as causes for stress on a watersheds vegetation. Unfortunately, the lower the vegetation-cover density in an area the more susceptible the earth will be to erosion from human and environmental stresses.

The ability of the soil to bind to the roots of plants growing on the surface is an important aspect of the model. The term vegetation-cover density or VCD has been mentioned previously and is defined as the percentage of ground area being covered by plants in a specific region. Naturally, a greater VCD will be desirable in the reduction of erosion. Since grasslands to do not provide a sufficient root base to effectively reduce erosion, we are only considering vegetation density of trees and shrubs. The VCD, which we will define simply as V for the model, is a very important indicator of the success of a watershed because it can be used to represent the level of vegetation development. The VCD can be affected drastically by many types of environmental and human occurrences and this fact makes its calculation rather complicated. For example, air pollution and drought can negatively affect an areas abilities in vegetation development. Though air pollution poses a much more enduring affect, a drought can cause a substantially greater, yet less lengthy, stress on the vegetation. If we define A_T as being the stress caused by air pollution, A as the amount of pollutant over a certain time interval, and A_a as the long term average of that same pollutant, we can identify the relationship $\frac{A-A_a}{A_a} = A_T$. Similarly, the precipitation in a given area can be defined as $\frac{P-P_e}{P_e} = P_T$ where P is the annual precipitation levels, P_e is the estimated water demand of a certain vegetated system, and P_T is the stress caused by a lack of precipitation in a watershed. Along with air pollution and precipitation, certain instantaneous events affect vegetation. Consider the meteorite that struck a Siberian forest in 1908 that flattened trees up to ten miles around the impact site. This instantaneous stress will be defined as $K_{inst}(t_0)$ or the instant stress at some time t_0 . Though humans can cause great stress on the environment through mining, road constructing, and farming activities, one human practice actually acts as one of the main deterrents of erosion and bolster to vegetation: the human reforestation effort. With continuity, the replanting of trees and shrubs in watersheds creates a strong and effective system that controls erosion. We can now construct a fairly thorough model for simulating vegetation development:

$$\frac{dV}{dt} = aV(1-V) - b\frac{EV}{V^2+1} + K_pP_T - K_aA_T - K_{inst}(t0) + V_R$$

(3) where E is the erosion density, a is a rate coefficients for how effectively vegetation begets more vegetation in the absence of erosion at low density, b gives a measure of how much the presence of erosion induces loss of vegetation, K_p and K_a represent a measure of the strength of the effects of air pollution and precipitation on a system, and V_R is a continuous function representing reforestation. In many case studies we find that the model can be reduced to a simpler form because there are only a few significant stresses on the system. A similar equation can be obtained which represents the erosion rate in watersheds.

$$\frac{dE}{dt} = cE(1-E) - d\frac{EV}{E^2+1} + E_R$$

where c is a rate coefficient for how effectively erosion begets more erosion in the absence of vegetation at low densities, d is a measure of how much the presence of vegetation impedes erosion, and E_R is a continuous function representing the influence of human activities on erosion.

By redefining the stresses imposed on vegetation due to environmental and human interference and the influences of humans on the erosion rate, we can produce a simpler set of differential equations that will be easier to manipulate. Let

$$V_T = K_pP_t - K_aA_t - K_{inst}(t0) + V_R$$

Substituting this new definition into our differential equations we produce the following set of equations:

$$\frac{dV}{dt} = aV(1-V) - b\frac{EV}{V^2+1} + V_T$$

$$\frac{dE}{dt} = cE(1-E) - d\frac{EV}{E^2+1} + E_R \ .$$

Note that we will assume $0 \leq E, V \leq 1$ throughout, as each of these represent fractions of area covered by either vegetation or erosion. Note however, that some patches of ground can have both erosion AND vegetation present together, so that we do not expect $E + V = 1$.

My intention for a group of three members is that each person be primarily responsible for one of problems 1,2, or 3, and that problem 4 be done as a group.

1. First we will consider our model in a natural setting away from human interference, so that $V_T = 0 = E_T$. In this case, do a full phase plane analysis, using the coefficients $a = .5$, $b = 1$, $c.7$, and $d = 1.4$. Give the equillibria and whether or not they appear to be stable or unstable and why. Discuss the meaning of what you find in your phase plane analysis in terms of the dynamics of the vegetation-erosion problem. What long-term outcomes are possible? Is there a deciding factor for whether or not erosion or vegetation wins long-term?

   Use MATLAB to sketch the solutions for at least three different sets of initial conditions taken from three different regions of the phase plane.

   Finally, change b so that $b = 0.5$. What would this change represent physically? Repeat the steps above. What differences do you see in the dynamics? Explain the differences you note in terms of what has changed in the physical system.

2. In problem 1, we look at a case where $\frac{a}{b} < 1$ and $\frac{a}{b} \geq 1$, with all other parameters held constant. Notice that in those cases, we also had $\frac{c}{d} < 1$. Let's explore how changing the ratio $\frac{c}{d}$ changes the dynamics of the system.

   Begin by setting $a = 0.5$ and $b = 1$, and let $c = 0.7$ and $d = 0.7$. Repeat the steps for problem 1 (creating a phase plane and using MATLAB to plot at least three different solutions over time). What differences do you see in the dynamics from what occured for the first part of problem 1? Explain the differences in terms of what has changed in the physical system.

   Next, let $c = 0.7$ and $d = 0.35$ keeping the values of $a = 0.5$ and $b = 1$. Again, repeat the steps for problem 1. Again, note any differences when compared to the work you've already done and explain them in terms of the changes in the physical system.

3. The Heishui River Basin is a river located near the Yangtze River in China. Because of its exceptionally high erosion rate and very low VCD, the Heishui River Basin is the perfect case in which to apply the model. Certain measures were taken to restore the vegetation cover density and relieve the stressful erosion levels. Trees in the area were reforested at a rate of five percent annually starting in 1978 and anti-erosion dams were constructed that reduced the crippling 7, 243 km/ton2 per year of original erosion by 650 km/ton2 per year. With an initial VCD of only 7.6 %, it would take a great deal of human interference to restore the river basin to an acceptable state. Because, in our case, the reforestation and anti-erosion efforts greatly outweigh the importance of other ecological stresses such as precipitation and air pollution, we can assume V_T and E_R are constant fractions of V and$E$. Let's set

$$V_T(t) = .05V \tag{1}$$

$$E_R(t) = -0.09E \tag{2}$$

Repeat the steps for problem 1, using the same parameter set as in problem 1 with $a = .5$, $b = 1$, $c = .7$ and $d = 1.4$ for the entire problem.

Suppose now that, in the opposite case, no such remediation measures are taken but that the environmental and human stresses on the ecosystem are present and constant and harmful, so

$$V_T(t) = -.1V \tag{3}$$

$$E_R(t) = 0.2E \tag{4}$$

Again, repeat the steps for problem 1, with the parameter values set as above.

What changes in the physical problem does this represent from what you had above? Do the changes in the outcomes agree with what you would expect to see?

4. Interpret your results from parts 1,2, and 3, in terms of the physical problem. Does it matter in what state the system starts out as far as what long-term outcomes you can expect? What seems to be the most important consideration in terms of maximizing the probability for a good outcome, if any? What would this lead you to conclude in terms of managing an ecosystem like this?

Analyze the phase plane you would get if you leave a, b, c and d undetermined in the origial set of equations ($V_T = 0$ and $E_R = 0$). Explain why whether or not $\frac{a}{b}$ and $\frac{c}{d}$ are greater than or equal to one or not is critical information for understanding what long term dynamics we are most likely to see play out.