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Math 231:Introduction to Ordinary Differential Equations
Mini-Project: Nuisance Beaver Trapping

Setting the Scene: Beavers, once hunted in open access for their pelts, were saved from extinction in the middle of this century by regulations controlling trapping season, method and numbers. Under this protection, the beaver population has rebounded in many regions of the country and has caused significant damage to valuable timber and agricultural land. Trapping is most effective in controlling beavers, whose primary nuisance is tree-cutting on privately-held timber land.

A trapping strategy that disregards the possible migratory behavior of beavers in neighboring "uncontrolled" (i.e., zero trapping) land parcels in filling the vacuum created by trapping in the "controlled" parcel, can be as futile in practice as attempting to dig a hole in fine-grain sand. We formulate a two-equation system of differential equations to model this phenomenon according to the recently formulated "social-fence" hypothesis of small mammal dispersion. This hypothesis can be viewed as the ecological analog of osmosis: Beavers from an environmentally superior habitat are posited to diffuse through a social fence to an inferior but less-densely populated habitat until the pressure to depart ("within-group aggression") is equalized with the pressure exerted against invasion ("between-group aggression"). This is termed "forward migration." Assuming that the controlled parcel is a superior habitat, the owner must be concerned with the "backward migration" that occurs when the superior parcel becomes less densely populated through trapping.

Rate Equations: Let X and Y represent nonnegative population densities [head/square mile] of beavers in the controlled and uncontrolled parcels respectively; and let X' and Y' represent the associated annual net rates of change [head/square mile/year]. The following pair of differential equations models X' and Y' as the difference between the rates of net growth (i.e., birth rate minus the death rate), dispersion, and, in the case of X, trapping:

$$X' = F_0(X)X - F_1(X,Y) - PX \tag{1}$$

$$Y' = F_2(Y)Y + F_1(X,Y) \tag{2}$$

where P [1/year] represents the per capita annual trapping rate of X. Thus, PX represents the total animals trapped each year.

$F_0(X)$ and $F_2(X)$ are logistic per capita (proportional) population growth rates for X and Y respectively with units [1/year] and are given by

$$F_0(X) = R_X(1 - X/K_X) \tag{3}$$

$$F_2(Y) = R_Y(1 - Y/K_Y) \tag{4}$$

where R_X [1/year], K_X [head/square mile], R_Y [1/year], and K_Y [head/square mile] are nonnegative constants. As the population density X approaches zero in the controlled parcel, the net proportional growth rate approaches R_X (i.e., $F_0 \to R_X$ as $X \to 0$), which is called the intrinsic growth rate [1/year]. Alternatively, as X approaches K_X, the net proportional growth rate decreases toward zero due to the negative impacts of crowding. Thus K_X is the environmental carrying capacity of the controlled parcel for beavers. The parameters R_Y and K_Y are interpreted analogously for the uncontrolled parcel. The total dispersion flux term, $F_1(X,Y)$, is a mathematical representation of the social-fence hypothesis (discussed above), which attempts to explain the dispersion of a small-mammal population between two adjacent parcels:

$$F_1(X,Y) = M(X/K_X - Y/K_Y)$$

where M and $-M$ [head/square mile/year] are the marginal disperse rates with respect to X/K_X and Y/K_Y, respectively. Whenever X constitutes a larger fraction of its carrying capacity than Y (i.e., $X/K_X > Y/K_Y$), within-group aggression of X is assumed to be greater than between-group aggression, and $F_1(X,Y)$ acts as a disperse valve allowing individuals to "forward" migrate from X to Y, i.e., $F_1(X,Y) > 0$. However, as trapping decreases the population pressure on carrying capacity in X, the disperse valve can become unidirectionally open for individuals to "backward" migrate from Y to X, i.e., $F_1(X,Y) < 0$.

The number of system parameters (i.e., R_X, R_Y, K_X, K_Y, M, P) involved in solving system (1)-(2) can be decreased from six to four by making the above quantities dimensionless. We do this by letting $x = X/K_X$ (the fraction of carrying capacity in the controlled parcel), $y = Y/K_Y$ (the fraction of carrying capacity in the uncontrolled parcel), and $\tau = R_X t$ (scaled time variable). Scaled parameters are $m = M/R_X K_X$ (the scaled dispersion parameter), $p = P/R_X$ (the scaled trapping parameter), $r = R_Y/R_X$ (the comparison of intrinsic growth rates in both patches), and $k = K_X/K_Y$ (the comparison of carrying capacities in both patches). We can use these new variables to show that the dimensionless model is

$$x' = \frac{dx}{d\tau} = x(1-x) - m(x-y) - px \tag{5}$$

$$y' = \frac{dy}{d\tau} = ry(1-y) + km(x-y) \tag{6}$$

For this project, and a group of three members, I intend for each member to be primarily responsible for one of the problems 1,2, or 3. Problems 4 and 5 are intended to be done as a group. It is important when making your conclusions that you remember that we do want to control the beaver populations, but we don't want to wipe them out completely.

1. Naturally-Regulated Dynamics: Consider first the "naturally-regulated" dynamics occurring when no trapping occurs in the timber-damaged parcel. This situation is modeled by setting p = 0 in (5). Let $k = 1$, $r = 1$, and $m = .75$. Do a phase plane analysis for this system and try to determine which equillibria are stable and which are unstable.

   Use MATLAB to find 3 different trajectories for 3 different initial conditions (use initial conditions from different regions to obtain qualitatively different results if possible). What do these results say about the physical behavior of these beaver populations over time?

   Repeat the above work with $r = 0.8$ and $k = 1.5$, keeping $m = 0.75$. What would this choice of parameters reflect physically versus or original choice of $k = 1$, $r = 1$? What would the difference be in what one would experience over time in each case?

2. Positive Trapping Rates: Consider now the impact of a nonzero trapping rate in the controlled parcel x. The system of differential equations governing the evolution of beaver populations when trapping occurs in the controlled parcel is given by system (5)-(6) where p is set at fixed rate. Assume further that p represents a 100% annual trapping rate (i.e., $P = 1$, so from collected data in the referred paper, $p = P/R_x = 2.985$), and that all other parameters are held at the previous values. Do the analysis for number 1 again with $p = 2.985$, and all other parameters are as described in problem 1.

3. What happens if the trapping is less efficient, so that $P = 0.5$ (50% of the beavers in the controlled area are trapped over one year). Find p and again analyze the system, making sure to again find at least three trajectories that begin in different regions of the phase plane. Comment on the physical meaning of your findings.

4. Finally, analyze what role the relative carrying capacity k has in whether or not trapping is effective. To do so, take the different values of $k = 0.25, 0.5, 2$, while holding $r = 0.8$, $m = 0.75$, and a 50% trapping rate constant.

   Find approximate solutions via MATLAB, with reasonable initial conditions for each of the three values of k and note the long term outcomes. What do you notice? What physically does k represent? Explain the difference in outcomes for the beaver populations in terms of how k is changing. What would this imply for a beaver management strategy?

5. What do your results from problems 1,2 3,and 4 tell us about whether or not trapping is a wise strategy for controlling the beaver population in the long term? Does it depend on the circumstances, or is it always a good idea? Summarize your results in terms of what the actual physical outcomes would be expected to be, and what sorts of things you would want to know about the controlled area and the uncontrolled area in order to make a wise trapping plan.

Citation: Huffaker, Bhat and Lenhart, "Optimal Trapping Strategies for Diffusing Nuisance-Beaver Populations," Natural Resource Modeling 6 (1992): 71-98.