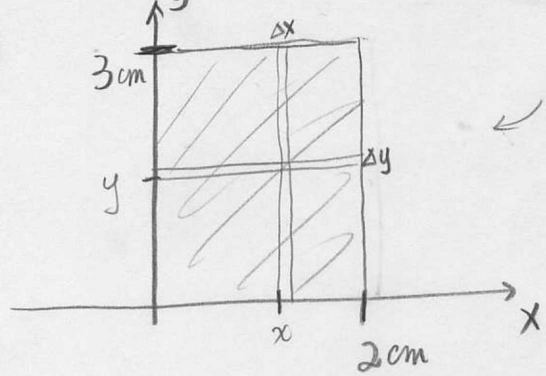
8.4: (a)



← plate has constant density $5\ \text{gm}/\text{cm}^2$

Finding mass with x-slices: Area of a strip $\cong 3\cdot\Delta x \Rightarrow$ mass of a strip $\cong 5\cdot 3\cdot \Delta x = 15\Delta x$

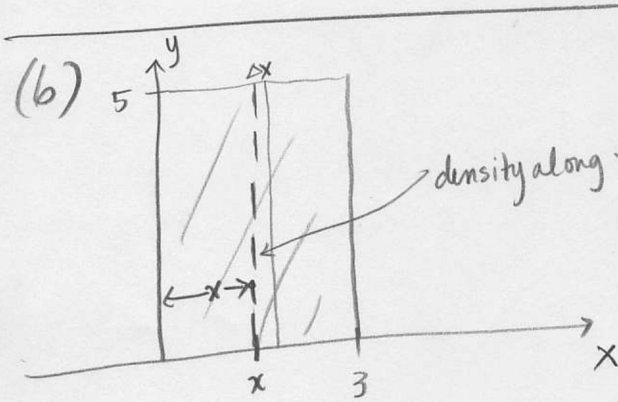
$$\text{total mass} = \int_0^2 15\,dx = 15x\Big|_0^2 = 30\ \text{gm}$$

Finding mass with y-slices: at a fixed y, the area of a strip is $\cong 2\Delta y$

$\Rightarrow$ mass of a strip $\cong 5\cdot 2\cdot \Delta y = 10\Delta y$

$$\text{total mass} = \int_0^3 10\,dy = 10y\Big|_0^3 = 30\ \text{gm}$$

(b)



density along this strip $= \dfrac{1}{1+x^4}$

$$\text{mass of strip} \cong \underbrace{\frac{1}{1+x^4}}_{\text{density}}\cdot\underbrace{5\Delta x}_{\text{area}} = \frac{5}{1+x^4}\Delta x$$

(a) $\quad \text{total mass} \cong \displaystyle\sum_{i=1}^{n} \frac{5}{1+x_i^4}\Delta x$

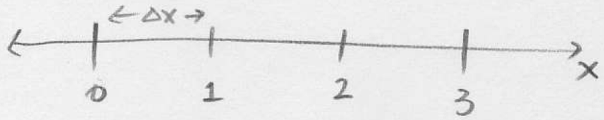
(b) Calculate the mass:

passing to the limit as $\Delta x \to 0$, we get mass $= \displaystyle\int_0^3 \frac{5}{1+x^4}\,dx$

BUT this is not integrable using our techniques, so we need to approximate it via Riemann sum.

(6 cont ...)

if I let $n=3$, $\quad \Delta x = \frac{3-0}{3} = 1$



the LHS $= \left(\frac{5}{1+0^4}\right)\Delta x + \left(\frac{5}{1+1^4}\right)\Delta x + \left(\frac{5}{1+2^4}\right)\Delta x$

$= 5\cdot 1 + \frac{5}{2}\cdot 1 + \frac{5}{17}\cdot 1 = \frac{170+85+10}{34} = \frac{265}{34} = 7.794$

the RHS $= \left(\frac{5}{1+1^4}\right)\Delta x + \left(\frac{5}{1+2^4}\right)\Delta x + \left(\frac{5}{1+3^4}\right)\Delta x$

$= \frac{5}{2}\cdot 1 + \frac{5}{17}\cdot 1 + \frac{5}{82}\cdot 1 \cong 2.5 + 0.294 + 0.061$

$= 2.855$

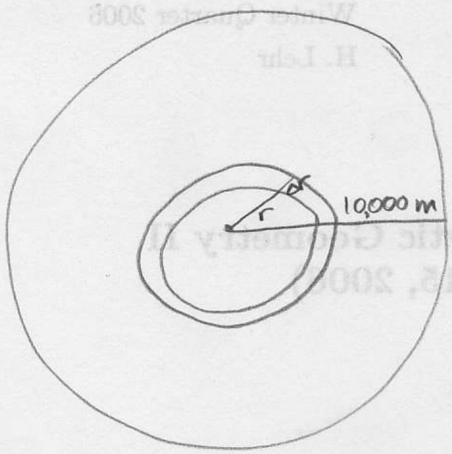
averaging the left + right approximations, we get

$$\int_0^3 \frac{5}{1+x^4}\,dx \cong \frac{7.794+2.855}{2} \cong 5.325$$

[$1/(1+x^4)$ is a strictly decreasing function, so the LHS is an overapprox, the right hand sum (RHS) is an under-estimate, and the average of the two will be a much better approximation]

(#10)

Circular oil slick



$\delta(r) = \frac{50}{1+r} \frac{kg}{m^2}$

$\uparrow$ density

(a) Since the density is constant along circles, our slices are circular.

$$\text{mass} \cong \sum \underbrace{(2\pi r \Delta r)}_{\text{area}} \cdot \underbrace{\frac{50}{1+r}}_{\text{density}}$$

(b) $$\text{mass} = \int_0^{10000} 2\pi r \cdot \frac{50}{1+r}\, dr = 100\pi \int_0^{10000} \frac{r}{1+r}\, dr$$

Let $u = 1+r \Rightarrow du = dr$ and $\Rightarrow r = u-1$

$$= 100\pi \int_1^{10001} \frac{u-1}{u}\, du$$

$$= 100\pi \int_1^{10001} 1 - \frac{1}{u}\, du = 100\pi \left[u - \ln|u|\right]_1^{10001}$$

$$= 100\pi \left[(10001 - \ln|10001|) - (1 - \cancel{\ln|1|}^{0})\right] = 100\pi\left[10000 - \ln|10001|\right]$$

(c) Within what distance $\hat{r}$ is half of the slick contained (by mass!)

$$\frac{1}{2}\text{mass} = 50\pi\left[10000 - \ln|10001|\right] = \int_0^{\hat{r}} 100\pi \cdot \frac{r}{1+r}\, dr = 100\pi\left[\hat{r}+1 - \ln|\hat{r}+1| - 1\right]$$

$$\Rightarrow 50\pi\left[10000 - \ln|10001|\right] = 100\pi\left[\hat{r} - \ln|\hat{r}+1|\right]$$

$$\frac{1}{2}\left[10000 - \ln|10001|\right] = \hat{r} - \ln|\hat{r}+1|$$

$$5000 - \ln(\sqrt{10001}) = \hat{r} - \ln|\hat{r}+1|$$

$\ln(\sqrt{10001}) \cong 4.605 \Rightarrow 5000 - 4.605 \cong \hat{r} - \ln|\hat{r}+1|$

$4995.4 \cong \hat{r} - \ln|\hat{r}+1|$

trying values of $\hat{r}$ near 5000:

try $\hat{r} = 5002 \Rightarrow \ln|\hat{r}+1| = \ln|5003| \cong 8.518$

so $\hat{r} - \ln|\hat{r}+1| \cong 5002 - 8.518 \cong 4993.5$

$\Rightarrow$ to small, so try slightly larger $\hat{r}$:

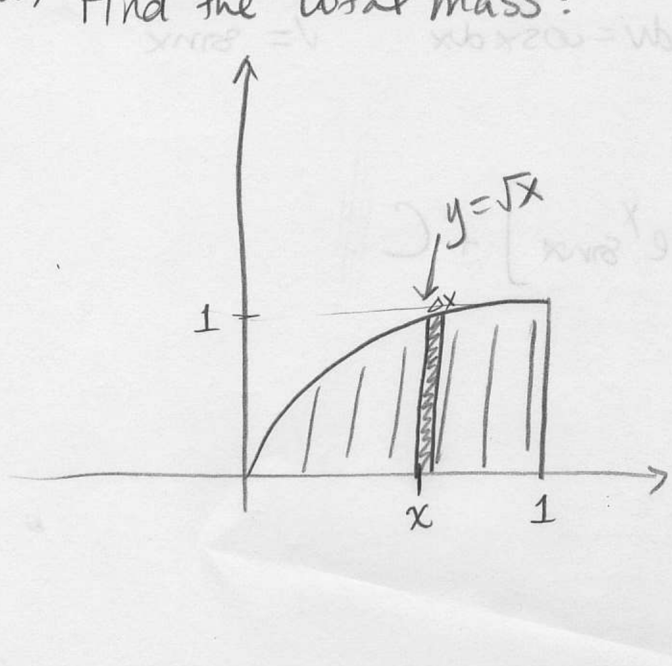
say $\hat{r} = 5004 \Rightarrow \ln|\hat{r}+1| \cong 8.518$

$\hat{r} - \ln|\hat{r}+1| \cong 5004 - 8.518 = 4995.5$

So an approximation for $\hat{r}$ is $\hat{r} = 5004$ meters and within 5004 meters, half the slick is contained.

(#24) A metal plate with (constant) density 5 gm/cm^2 is bounded by $y=\sqrt{x}$, the x-axis and $0 \le x \le 1$.

(a) Find the total mass:



mass of an x-slice $\cong$ area $\times$ density $\cong \sqrt{x}\,\Delta x \cdot 5$

$\Rightarrow$ total mass $= \int_0^1 5\sqrt{x}\,dx$

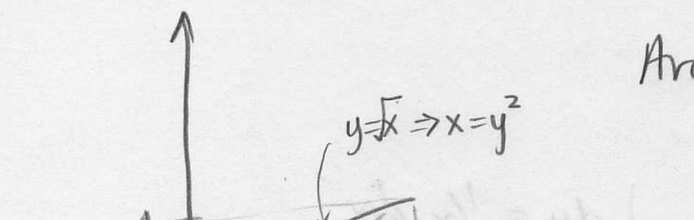
$= \frac{5x^{3/2}}{3/2}\Big|_0^1 = \frac{5}{3/2} = \frac{10}{3}$ gm

(b) Find $\bar{x}$ and $\bar{y}$ (where $(\bar{x}, \bar{y})$ is the center of mass)

$$\bar{x} = \frac{\int_0^1 5\cdot x\sqrt{x}\,dx}{\int_0^1 5\sqrt{x}\,dx} = \frac{\int_0^1 5x^{3/2}dx}{10/3} = \frac{\frac{5x^{5/2}}{5/2}\Big|_0^1}{10/3}$$

$$= \frac{2}{10/3} = \frac{6}{10} = \frac{3}{5}$$

$$\bar{y} = \frac{\int_0^1 5\cdot y\cdot(\text{area of a } y\text{-strip})}{10/3}$$



Area of a y-strip:

$\Delta y \cdot (1-y^2)$

$$\bar{y} = \frac{\int_0^1 5y(1-y^2)\,dy}{10/3}$$

$$= \frac{\int_0^1 (5y - 5y^3)\,dy}{10/3}$$

$$= \frac{\frac{5}{2}y^2 - \frac{5}{4}y^4\Big|_0^1}{10/3} = \frac{5/4}{10/3} = \frac{15}{40} = \frac{3}{8}$$

$\Rightarrow$ the center of mass is $(\bar{x}, \bar{y}) = (3/5, 3/8)$