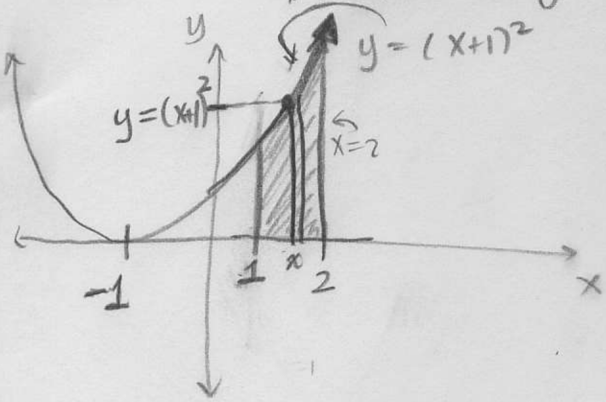
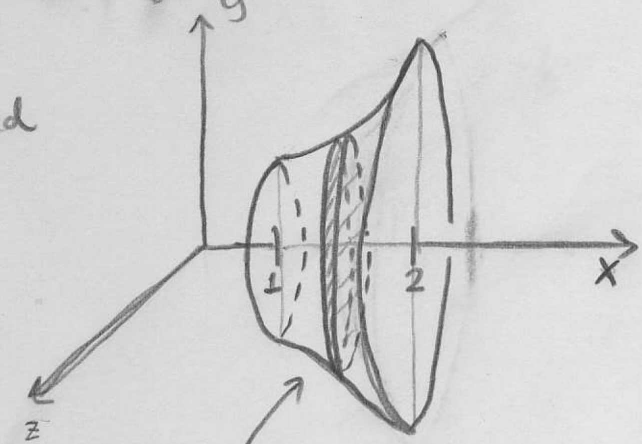
8.2 :

(#2) Find the volume of the region rotation about the x-axis :

bounded by $y=(x+1)^2$, $y=0$, $x=1$, $x=2$



rotated $\Longrightarrow$



if we slice vertically (x-slices), our slices are disks.

The volume of a slice is:

$\cong \pi(\underbrace{(x+1)^2}_{r})^2 \Delta x$

$\Rightarrow$ Volume of solid $= \displaystyle\int_1^2 \pi (x+1)^4 dx$

$r=(x+1)^2$ because this is this dist. from the axis of rotation to the outer edge of the solid — or to the curve $y=(x+1)^2$ looking at the cross-section in the x-y plane

Let $u=x+1 \Rightarrow du=dx$, and $= \displaystyle\int_2^3 \pi u^4 du = \frac{\pi u^5}{5}\Big|_2^3 = \frac{\pi}{5}\left(3^5-2^5\right)$

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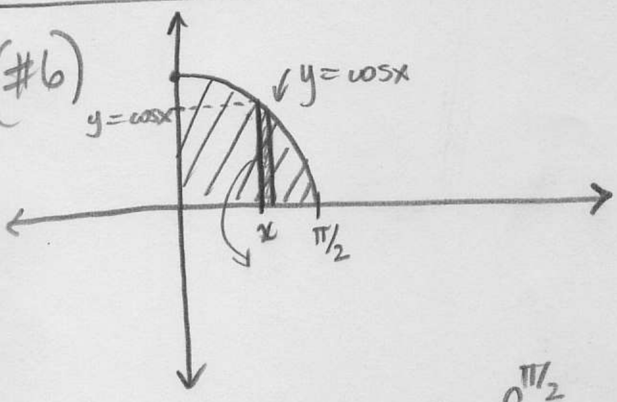
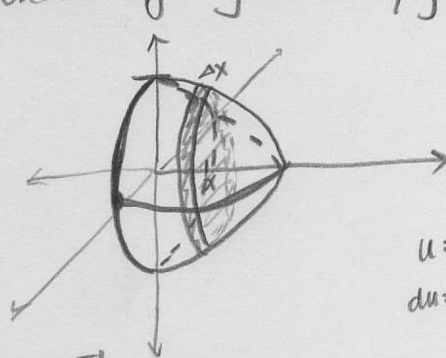
bounded by $y=\cos x$, $y=0$, $x=0$, $x=\pi/2$

(#6)



$\Rightarrow$



Again, vertical slices (x-slices) are disks!

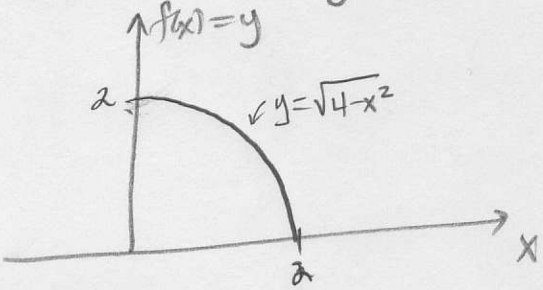
$u=\cos x$ $\quad dv=\cos x\,dx$

$du=-\sin x$ $\quad v=\sin x$

$\text{Volume} = \displaystyle\int_0^{\pi/2} \pi(\cos x)^2 dx = \pi \int_0^{\pi/2} \cos x \cdot \cos x\, dx = \pi\left[\cos x \sin x\Big|_0^{\pi/2} + \int_0^{\pi/2} \underbrace{\sin^2 x}_{1-\cos^2 x} dx\right]$

$= \pi\left[\cos x\sin x + x\Big|_0^{\pi/2} - \displaystyle\int_0^{\pi/2} \cos^2 x\,dx\right] \Rightarrow \pi\int_0^{\pi/2}\cos^2 x\,dx = \frac{\pi}{2}\left[(0+\pi/2)-0\right]$

#12: arclength of $f(x)=\sqrt{4-x^2}$ from $x=0$ to $x=2$:



$$f'(x) = \frac{1}{2}(4-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{4-x^2}}$$

$$\text{Arclength} = \int_0^2 \sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\,dx$$

$$= \int_0^2 \sqrt{\frac{4-x^2}{4-x^2}+\frac{x^2}{4-x^2}}\,dx = \int_0^2 \sqrt{\frac{4}{4-x^2}}\,dx$$

$$= \int_0^2 \frac{2}{\sqrt{4-x^2}}\,dx \qquad \text{let } x = 2\sin\theta \Rightarrow dx = 2\cos\theta\,d\theta$$

$$= \int_0^2 \frac{2}{\sqrt{4-4\sin^2\theta}}\cdot 2\cos\theta\,d\theta = \int_0^2 \frac{4\cos\theta}{\sqrt{4}\underbrace{\sqrt{1-\sin^2\theta}}_{=\cos\theta}}\,d\theta$$

$$= \frac{4}{2}\int_0^2 d\theta = 2\theta\Big|_0^2 = 2\arcsin\left(\frac{x}{2}\right)\Big|_0^2$$

$$= 2\big[\underbrace{\arcsin(1)}_{=\pi/2} - \underbrace{\arcsin(0)}_{=0}\big] = 2\cdot\frac{\pi}{2} = \pi$$

#14: parametric curve given by $x(t) = 3+5t$, $y(t) = 1+4t$ $(1 \le t \le 2)$

$$\text{arclength} = \int_1^2 \sqrt{(5)^2+(4)^2}\,dt = \int_1^2 \sqrt{25+16}\,dt = \int_1^2 \sqrt{41}\,dt = \sqrt{41}\int_1^2 dt$$

$$= \sqrt{41}\left(t\Big|_1^2\right) = \sqrt{41}$$

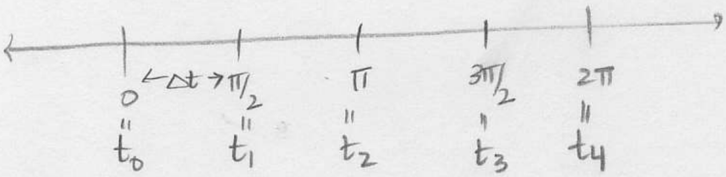
\# 16: $x(t) = \cos(3t)$, $y = \sin(5t)$

find the arclength of the curve for $0 \le t \le 2\pi$

$$\text{arclength} = \int_0^{2\pi} \sqrt{(-3\sin(3t))^2 + (5\cos(5t))^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{9\sin^2(3t) + 25\cos^2(5t)}\, dt$$

(this is not integrable via our techniques – our only recourse is to approximate it using Reimann sums!)



I'm taking $n=4 \Rightarrow \Delta t = \frac{2\pi - 0}{4} = \frac{\pi}{2}$, and let $f(t) = \underbrace{\sqrt{9\sin^2(3t) + 25\cos^2(5t)}}_{\text{integrand}}$

$$\text{LHS} = f(t_0)\Delta t + f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t$$

$$= f(0)(\pi/2) + f(\pi/2)(\pi/2) + f(\pi)\cdot \pi/2 + f(\tfrac{3\pi}{2})\cdot \pi/2$$

$$= \sqrt{25}\,(\pi/2) + \sqrt{9}\,(\pi/2) + \sqrt{25}\cdot \pi/2 + \sqrt{9}\cdot \pi/2 = 3\pi + 5\pi = 8\pi$$

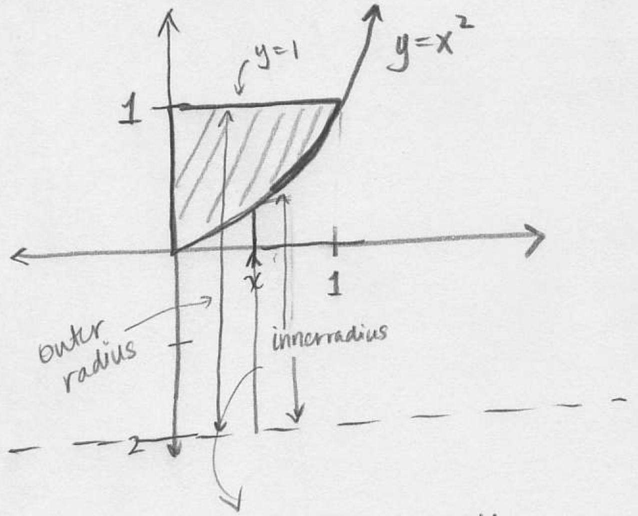
$$\text{the RHS} = f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t + f(t_4)\Delta t$$

$$= \sqrt{9}\,(\pi/2) + \sqrt{25})(\pi/2) + \sqrt{9}\cdot \pi/2 + \sqrt{25}\cdot \pi/2 = 8\pi$$

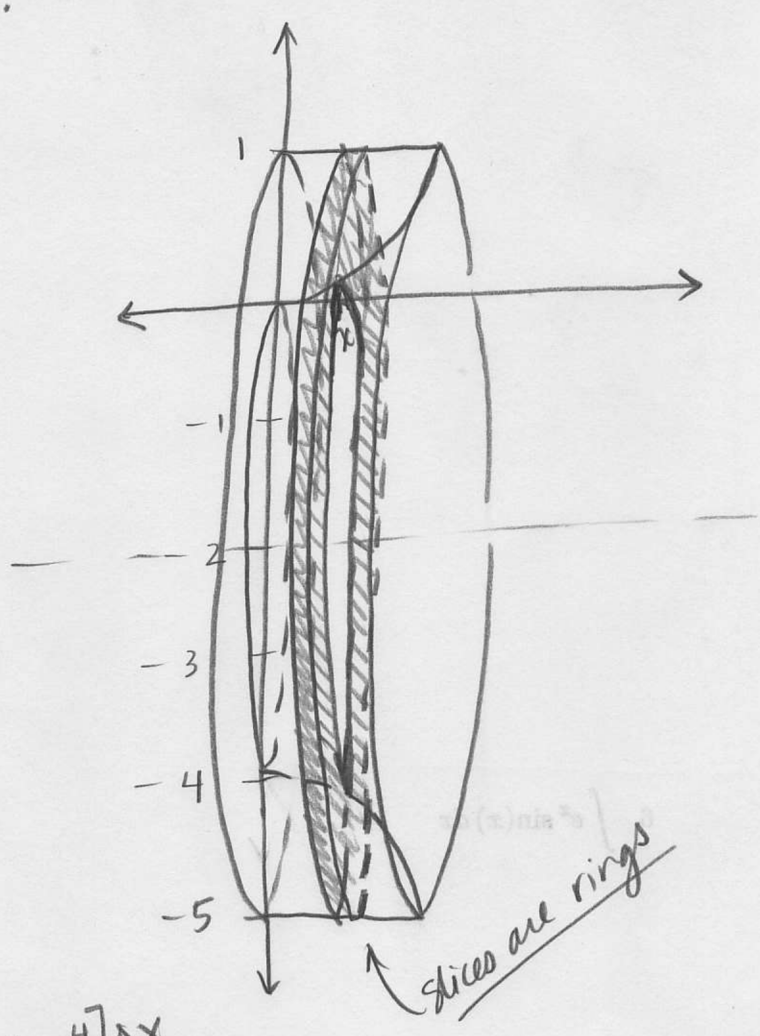
$$\Rightarrow \text{the average is } \frac{\text{LHS} + \text{RHS}}{2} = 8\pi \cong \int_0^{2\pi} \sqrt{9\sin^2(3t) + 25\cos^2(5t)}\, dt$$

$\hookrightarrow$ [Notice: averaging in this case did not improve our approximation. The integrand is not strictly increasing or strictly decreasing?...]

\# 24) Find the volume of the solid obtained by rotating the region bdd by $y=x^2$, $y=1$, and the y-axis around the line $y=-2$.



$\Rightarrow$



our slices are rings with width Δx, inner radius $= x^2-(-2) = x^2+2$

outer radius $= 1-(-2) = 3$

So the volume of a slice

$\cong$ slice area $\times \Delta x$

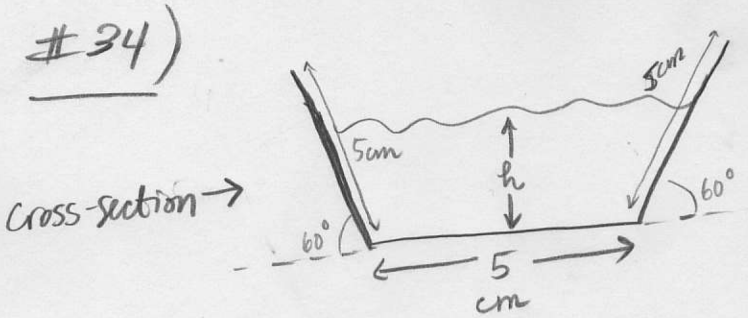
$= \left(\pi (3)^2 - \pi (x^2+2)^2\right)\Delta x$

$= \pi \left[9 - (x^4+4x^2+4)\right]\Delta x = \pi\left[5-4x^2-x^4\right]\Delta x$

volume of solid $= \displaystyle\int_0^1 \pi(5-4x^2-x^4)\,dx = \pi\left[5x-\frac{4}{3}x^3-\frac{1}{5}x^5\right]_0^1$

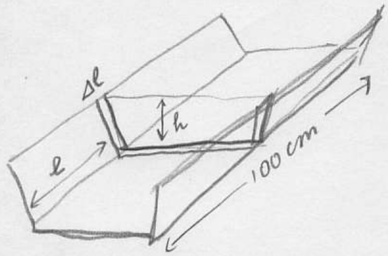
$= \pi\left[5-4/3-1/5\right] = \pi\left[\frac{25}{15}-\frac{20}{15}-\frac{3}{15}\right] = \frac{2\pi}{15}$

34)



A 100 cm long gutter has it's cross-section as shown.

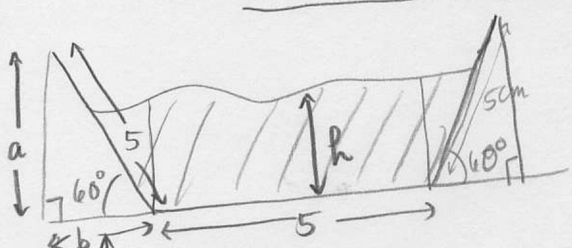
(a) find the volume of water in the gutter for a water depth of h cm.



each slice looks like. Let l = distance from front end of the gutter.

then the volume of water in a slice is given by $\Delta l \times$(sfce area)

What's the surface area of a slice?



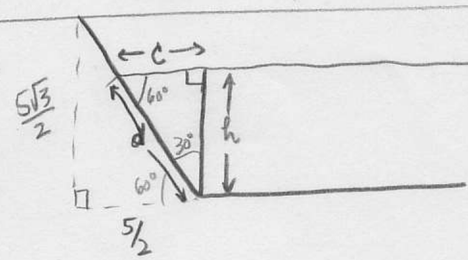
by trig, $\sin 60° = \frac{a}{5}$ and $\cos 60° = \frac{b}{5}$

$\Rightarrow a = 5\sin 60° = 5 \cdot \frac{\sqrt{3}}{2}$ and $b = \cos 60° \cdot 5 = 5 \cdot \frac{1}{2}$

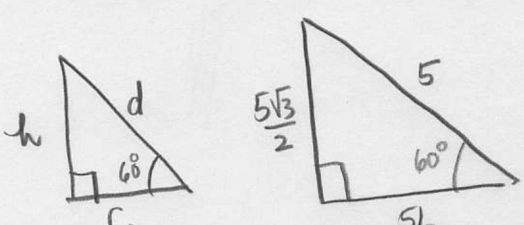
$a = \frac{5\sqrt{3}}{2}$, $b = 5/2$

---

we can now look at the left end of the gutter:



and we have similar triangles



$\Rightarrow \frac{h}{5\sqrt{3}/2} = \frac{d}{5}$ and $\frac{h}{\frac{5\sqrt{3}}{2}} = \frac{c}{5/2}$

$\Rightarrow d = \frac{2h}{\sqrt{3}}$ and $c = \frac{h}{\sqrt{3}}$

So the area of a slice is:

$2 \times \left[\left(\frac{1}{2}\right)(h)\left(\frac{h}{\sqrt{3}}\right)\right] + 5h = \frac{h^2}{\sqrt{3}} + 5h$

Volume of water in a slice $\cong \left(\frac{h^2}{\sqrt{3}} + 5h\right)\Delta l$

$$\Rightarrow \text{total volume of water} = \int_0^{100} \left(\frac{h^2}{\sqrt{3}} + 5h\right) dl = \left(\frac{h^2}{\sqrt{3}} + 5h\right) \int_0^{100} dl$$

$$= 100 \cdot \left(\frac{h^2}{\sqrt{3}} + 5h\right)$$

this is constant with respect to l!

(b) What's the max value of h?

we found this in part (a) $\rightarrow$ $h_{max} = \frac{5\sqrt{3}}{2}$ cm

(c) $$\text{Max volume of water} = 100 \cdot \left(\frac{\left(\frac{5\sqrt{3}}{2}\right)^2}{\sqrt{3}} + 5\left(\frac{5\sqrt{3}}{2}\right) \right) (cm^3)$$

$$= 100\left(\frac{25}{4}\sqrt{3} + \frac{25}{2}\sqrt{3}\right) = \frac{7500}{4}\sqrt{3} \text{ cm}^3$$

(d) + (e) if vol of water $= \frac{1}{2}$ max volume $= \frac{7500}{8}\sqrt{3}$ cm^3

is the depth greater or smaller than $5\sqrt{3}/4 = \frac{1}{2}$ max height?

$$\text{volume} = 100\left(\frac{h^2}{\sqrt{3}} + 5h\right) = \frac{7500}{8}\sqrt{3} \Rightarrow h^2 + 5\sqrt{3}h = \frac{75}{8} \cdot 3 = \frac{225}{8}$$

$$\Rightarrow h^2 + 5\sqrt{3}h - \frac{225}{8} = 0 \Rightarrow h = \frac{-5\sqrt{3} \pm \sqrt{75 + \frac{225}{2}}}{2}$$

$$= \frac{-5\sqrt{3} \pm \sqrt{\frac{375}{2}}}{2}$$

$$\cong \frac{-8.66 \pm 13.69}{2} \simeq \frac{5}{2}, \text{ or a negative \#}$$

So the question is, is $\underset{\overset{\parallel}{2.5}}{\frac{5}{2}} > \frac{5\sqrt{3}}{4} \cong \frac{8.66}{4} = 2.33$

yes! So if the gutter has half the max volume, the height of the water is greater than half the max height.