1. You are offered two jobs starting on June 1^{st} of 2006. Firm A offers you \$40,000 a year to start and you can expect a raise of 4% each June 1^{st}. Firm B has a starting salary of \$30,000 a year, but has an expected annual raise of 6%.

(a) [4 points] Give two functions: $f(t)$ describing the salary as a function of years at firm A, and $g(t)$ giving the salary as a function of years at firm B.

using the exponential growth model (because there is a constant percentage increase per unit time) $P = P_0 a^t$

with P_0 = starting salary = $P(0)$

and $a = 1 +$ percentage increase per year

with t = # of years

$$f(t) = 40000(1.04)^t$$
$$g(t) = 30000(1.06)^t$$

[OR $f(t) = 60000(1.05)^t$
$g(t) = 50000(1.07)^t$
for version A of exam]

(b) [4 points] How long it would take for the salary at firm A and firm B to be the same?

This would be the time t at which $f(t) = g(t)$:

$$40000(1.04)^t = 30000(1.06)^t$$

Solving for t:

$$\frac{4}{3}(1.04)^t = (1.06)^t$$

$$\Rightarrow \log\left(\frac{4}{3}(1.04)^t\right) = \log(1.06)^t$$

$$\Rightarrow \log\left(\frac{4}{3}\right) + \log(1.04)^t = \log(1.06)^t$$

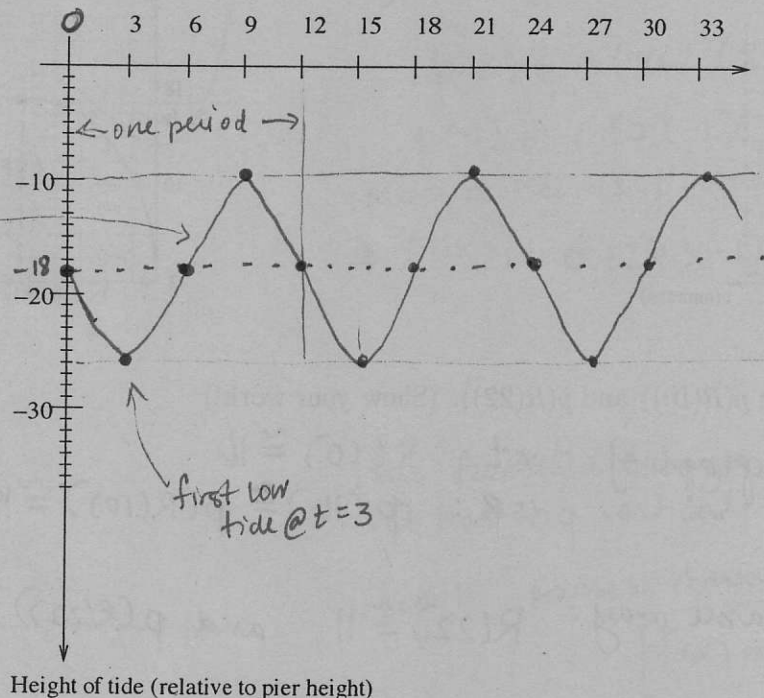
$$\log(4/3) + t\log(1.04) = t\log(1.06)$$

$$\Rightarrow t\left[\log(1.04) - \log(1.06)\right] = -\log(4/3)$$

$$t = \frac{-\log(4/3)}{\log(1.04) - \log(1.06)}$$

2. At high tide, the water level is 10 feet below a certain pier. At low tide the water level is 26 feet below the pier.

(a) [4 points] Assuming sinusoidal behavior, sketch a graph of $y = f(t)$ describing the water level, relative to the pier, at time t, if at $t = 0$ the water level is -18 feet and falling until the first low tide is reached at $t = 3$.



high tide height = −10 ft

[average of −10 + −26 is −18 ft ⇒ this is the axis of oscillation

low tide height = −26 ft

we can guess these because we know the curve is sinusoidal!

(b) [2 points] Find the period of $f(t)$.

the tide passes from mid tide to low tide in 3 units of time, so since it behaves sinusoidally, we can guess that after 3 more units it returns to mid tide and so on.
According to this and the resulting graph the period would be 12 units of time = amt of time required to repeat itself.

(c) [2 points] Find the amplitude of $f(t)$.

$$\text{Amp} = \frac{\text{low tide} - \text{high tide}}{2} = \frac{-10+26}{2} = \frac{16}{2} = 8 \text{ ft}$$

[for version A, amp = 4 ft]

(d) [4 points] Based on your sketch and the information provided, give a formula for $f(t)$.

for version A:
A=4
D=−9
all else is same!

$$f(t) = A\sin(B(t+C)) + D$$

$$B = \frac{2\pi}{\text{newperiod}} = \frac{2\pi}{12} = \frac{\pi}{6}$$

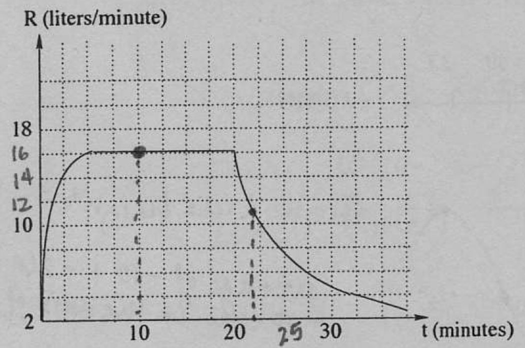
$A = \text{Amp} = 8$

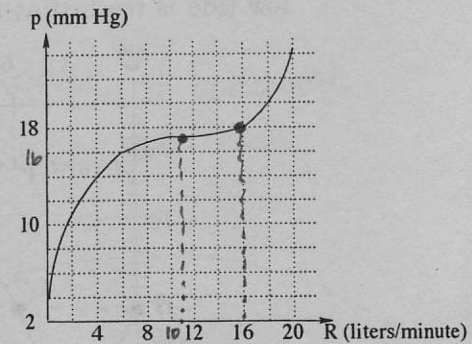
$C = \text{horiz. shift} = \text{right by } 6$

$D = \text{vertical shift} = -18$

$$f(t) = 8\sin\left(\frac{\pi}{6}(t - 6)\right) - 18$$

3. One of the graphs below shows the rate of flow, R, of blood from the heart in a man who bicycles for twenty minutes, starting at $t = 0$ minutes. The other graph shows his blood pressure, p, as a function of the rate of flow of blood from the heart.





(a) [4 points] Estimate $p(R(10))$ and $p(R(22))$. (Show your work!)

By the graph of R vs t: $R(10) \cong 16$

" " " " p vs R: $p(16) \cong p(R(10)) \cong 18$

in the same way: $R(22) \cong 11$ and $p(R(22)) \cong p(11) \cong 17$

(b) [2 points] Explain what $p(R(10))$ represents in practical terms.

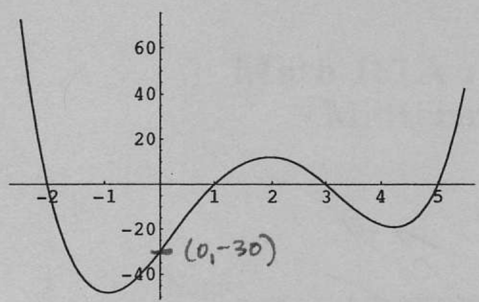
$p(R(10))$ is the man's blood pressure after riding the bike for 10 minutes.

(c) [2 points] Is the composite function $f(t) = p(R(t))$ invertible? Explain.

for $p(R(t))$ to be invertible every output value must be uniquely associated to an input. But if his blood pressure is 18 mmHg, we cannot determine exactly what time it is! It could be any time t between 6 and 20 because if $6 \le t \le 20$, then $R(t) \cong 16$ and $p(R(t)) \cong 18$

4. For each of the graphs below, find a possible formula for the function. In (c), assume the numerator is a linear function.

(a) [3 points]



This could be a polynomial with zeros at $x = -2, 1, 3,$ and 5:

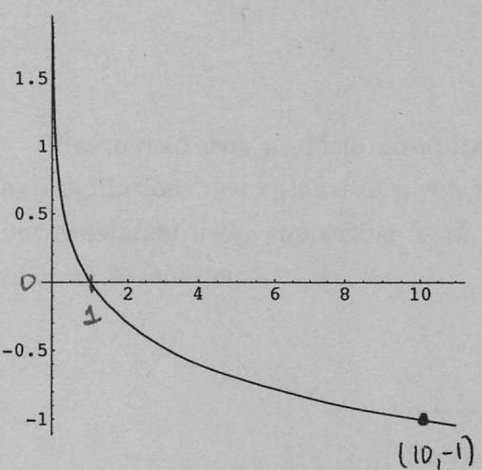
$$p(x) = c\,(x+2)(x-1)(x-3)(x-5)$$

$$p(0) = c\,(2)(-1)(-3)(-5) = -30 \cdot c$$

since it seems $p(0) = -30$, $c = 1$ is fine.

$$\Rightarrow p(x) = (x+2)(x-1)(x-3)(x-5)$$

(b) [3 points]



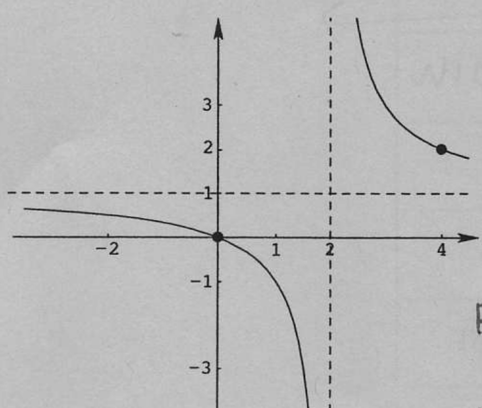
This function changes slowly as x grows, but has no vertical asymptote $\Rightarrow$ logarithmic

$\log_{10} x$ passes through $(10, 1)$, so let

$$\boxed{f(x) = -\log_{10}(x)}$$

Check: $f(1) = -\log_{10}(1) = 0$ ✓
$f(10) = -\log_{10}(10) = -1$ ✓

(c) [3 points]



Rational function because it has horiz. and vert. asymptotes. In the directions, it says the numerator is linear, so $g(x) = \dfrac{ax+b}{?}$

There is a vertical asymptote at $x = 2 \Rightarrow$ let denominator be $x - 2$: $g(x) = \dfrac{ax+b}{x-2}$

By graph $g(0) = 0$: $g(0) = \dfrac{a \cdot 0 + b}{-2} = \dfrac{b}{-2} = 0 \Rightarrow b = 0$

and $g(4) = 2$: $g(4) = \dfrac{a \cdot 4}{4-2} = \dfrac{4a}{2} = 2 \Rightarrow a = 1$

so $\boxed{g(x) = \dfrac{x}{x-2}}$

(d) [3 points] Describe the continuity, concavity, and increasing/decreasing properties of the function shown in part(c).

- g is continuous everywhere except at $x = 2$.
- g is concave down on $(-\infty, 2)$ and " up on $(2, \infty)$
- g is decreasing everywhere [except at $x = 2$ because it is undefined here]