Ex1:

$$\int \frac{z^3}{\sqrt{9-z^2}}\,dz \qquad \text{let } u = 9-z^2 \Rightarrow z^2 = 9-u$$

$$du = -2z\,dz$$

$$-\tfrac{1}{2}du = z\,dz$$

$$= \int \frac{z^2}{\sqrt{9-z^2}} \cdot z\,dz = \int \frac{9-u}{\sqrt{u}} \cdot \left(-\tfrac{1}{2}\right) du$$

$$= -\tfrac{1}{2} \int u^{-1/2}(9-u)\,du = -\tfrac{1}{2} \int 9u^{-1/2} - u^{1/2}\,du$$

$$= -\tfrac{1}{2}\left[\frac{9u^{1/2}}{1/2} - \frac{u^{3/2}}{3/2}\right] + C$$

$$= -9\sqrt{9-z^2} + \tfrac{1}{3}(9-z^2)^{3/2} + C$$

Ex2:

$$\int \frac{dx}{(x^2+9)^3} \qquad \text{since } 9\tan^2\theta + 9 = 9\sec^2\theta$$

$$\text{let } x = 3\tan\theta$$

$$\Rightarrow dx = 3\sec^2\theta\,d\theta$$

$$= \int \frac{3\sec^2\theta}{(9\tan^2\theta+9)^3}\,d\theta$$

$$= \int \frac{3\sec^2\theta}{(9\sec^2\theta)^3}\,d\theta = \frac{3}{9^3}\int \frac{1}{\sec^4\theta}\,d\theta = \frac{3}{9^3}\int \cos^4\theta\,d\theta$$

$$= \frac{3}{9^3}\int \left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta = \frac{3}{4\cdot 9^3}\int 1 + 2\cos(2\theta) + \cos^2(2\theta)\,d\theta$$

$$= \frac{3}{4\cdot 9^3}\left[\theta + \sin(2\theta) + \int \cos^2(2\theta)\,d\theta\right]$$

$$= \frac{3}{4\cdot 9^3}\left[\theta + \sin(2\theta) + \frac{1}{2}\int (1+\cos(4\theta))\,d\theta\right]$$

$$= \frac{3}{4\cdot 9^3}\left[\theta + \sin(2\theta) + \frac{1}{2}\left[\theta + \frac{\sin(4\theta)}{4}\right]\right] + C$$

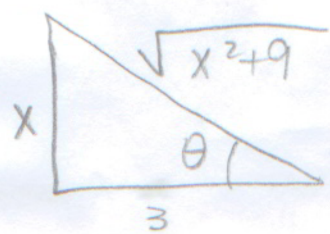
since $\sin(2\theta) = 2\sin\theta\cos\theta$

$$\hookrightarrow = \frac{3}{4\cdot 9^3}\left[\theta + 2\sin\theta\cos\theta + \frac{1}{2}\left[\theta + \frac{2}{4}\sin(2\theta)\cos(2\theta)\right]\right] + C$$

+ since $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$$\hookrightarrow = \frac{3}{4\cdot 9^3}\left[\frac{3\theta}{2} + 2\sin\theta\cos\theta + \frac{1}{4}\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)\right] + C$$

we had: $\frac{x}{3} = \tan\theta \Rightarrow$



so

$$= \frac{3}{4\cdot 9^3}\left[\frac{3}{2}\arctan\left(\frac{x}{3}\right) + 2\left(\frac{x}{\sqrt{x^2+9}}\right)\left(\frac{3}{\sqrt{x^2+9}}\right) + \frac{1}{4}\left(\frac{x}{\sqrt{x^2+9}}\right)\left(\frac{3}{\sqrt{x^2+9}}\right)\left(\frac{9}{x^2+9} - \frac{x^2}{x^2+9}\right)\right] + C$$