Name: KEY #2

Each problem is worth 30 points. Work at least 10 problems; you may work one additional problem as a bonus problem worth 15 points. Be sure to indicate which problem is to be graded as a bonus; if you fail to do so I will grade the first 10 problems you work out of 30 and the eleventh out of 15. If you work all 12 I will only grade the first 11. Show all your work for full credit; numerical or graphical estimates are unacceptable unless specifically requested. Use of calculators with symbolic algebraic capability such as the TI-89 is prohibited and will result in a grade of zero on this exam.

1. Use the definition of the integral (NOT FTC) to evaluate

$$\int_0^3 x^3 \mathrm{d}x$$

(Hint: Use the fact that $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.)

$$= \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f(x_k) = \lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} \left(\frac{3k}{n}\right)^3$$

$$= \lim_{n\to\infty} \frac{81}{n^4} \sum_{k=1}^{n} k^3 = \lim_{n\to\infty} \frac{81}{n^4} \frac{n^2(n+1)^2}{4}$$

$$= \lim_{n\to\infty} \frac{81}{4} (1)^2 (1+1/n)^2 = 81/4 = 20.25$$

2. Find

(a)

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \sqrt{\sec \ln \frac{t^2+3}{t}}\mathrm{d}t$$

(b)

$$\int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{\sec \ln \frac{x^2+3}{x}}\right)\mathrm{d}x$$

(a) $\sqrt{\sec \ln \frac{x^2+3}{x}}$

(b) $\sqrt{\sec \ln \frac{b^2+3}{b}} - \sqrt{\sec \ln \frac{a^2+3}{a}}$

3. Find the indefinite integral:

(a) $\int 5^x \mathrm{d}x$

(b) $\int -\frac{1}{\sqrt{1-t^2}} \mathrm{d}t$

(c) $\int \underset{u}{x^3} \underset{dv}{e^x} \mathrm{d}x$

(d) $\int x^2 e^{(x^3)} \mathrm{d}x$

(e) $\int \cos^2 \theta \mathrm{d}\theta$

(a) $5^x/\ln 5 + C$

(b) $\cos^{-1} t + C$

(c) $= x^3 e^x - \int \underset{u}{3x^2} \underset{dv}{e^x} dx$

$= x^3 e^x - \left(3x^2 e^x - \int \underset{u}{6x} \underset{dv}{e^x} dx \right)$

$= x^3 e^x - 3x^2 e^x + \left(6x e^x - \int 6 e^x dx \right)$

$= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C$

(d) $= \frac{1}{3} \int 3x^2 e^{x^3} dx = \frac{1}{3} e^{x^3} + C$

(e) $= \int \frac{1 + \cos 2\theta}{2} = \frac{\theta}{2} + \frac{1}{4} \sin 2\theta + C$

4. Find the indefinite integral:

$$\int \sqrt{1-x^2}\mathrm{d}x$$

$x = \sin\theta$

$dx = \cos\theta \, d\theta$

$\theta = \sin^{-1}x$

$\cos\theta = \sqrt{1-\sin^2\theta}$

$\cos\sin^{-1}x = \sqrt{1-x^2}$

$$= \int \sqrt{1-\sin^2\theta}\,\cos\theta\, d\theta$$

$$= \int \sqrt{\cos^2\theta}\,\cos\theta\, d\theta$$

$$= \int \cos^2\theta\, d\theta$$

$$= \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + c$$

$$= \frac{\theta}{2} + \frac{1}{2}\sin\theta\cos\theta + c$$

$$= \frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1-x^2} + C$$

5. Find the indefinite integral:

$$\int \frac{3x^3 + x^2 - 2x + 4}{x^4 + x^2} dx$$

$$\frac{3x^3 + x^2 - 2x + 4}{x^4 + x^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x^2} + \frac{D}{x}$$

$$3x^3 + x^2 - 2x + 4 = (Ax + B)x^2 + C(x^2 + 1) + Dx(x^2 + 1)$$

$$= Ax^3 + Bx^2 + Cx^2 + C + Dx^3 + Dx$$

$$= (A + D)x^3 + (B + C)x^2 + Dx + C$$

$A + D = 3$ $\qquad A = 5$

$B + C = 1$ $\qquad B = -3$

$-2 = D$

$4 = C$

$$\int \left(\frac{5x - 3}{x^2 + 1} + \frac{4}{x^2} - \frac{2}{x} \right) dx$$

$$= \frac{5}{2} \int \frac{2x}{x^2 + 1} dx - 3 \int \frac{1}{x^2 + 1} dx + 4 \int x^{-2} dx - 2 \int \frac{1}{x} dx$$

$$= \frac{5}{2} \ln|x^2 + 1| - 3 \tan^{-1} x - \frac{4}{x} - 2 \ln|x| + C$$

6. Suppose the velocity of a particle at time t is given by $v(t) = t^3 - 3t^2 + 2t$.

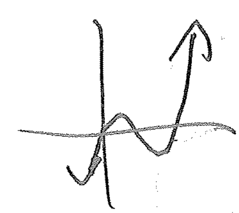
(a) Find the displacement of the particle after 10 seconds, assuming it starts at the origin.

(b) Find the distance traveled by the particle during the first 10 seconds.

(a) $$\int_0^{10} (t^3 - 3t^2 + 2t)\,dt = \frac{t^4}{4} - t^3 + t^2 \Big]_0^{10}$$

$$= 2500 - 1000 + 100 = 1600$$

(b) $$0 = t^3 - 3t^2 + 2t$$
$$= t(t^2 - 3t + 2)$$
$$= t(t-2)(t-1)$$
$$t = 0, 1, 2$$



$$\int_0^1 (t^3 - 3t^2 + 2t)\,dt - \int_1^2 (t^3 - 3t^2 + 2t)\,dt + \int_2^{10} (t^3 - 3t^2 + 2t)\,dt$$

$$= \left[\frac{t^4}{4} - t^3 + t^2\right]_0^1 - \left[\frac{t^4}{4} - t^3 + t^2\right]_1^2 + \left[\frac{t^4}{4} - t^3 + t^2\right]_2^{10}$$

$$= \left(\left(\frac{1}{4} \cancel{-1+1}\right) - 0\right) - \left(\left(\cancel{4-8+4}\right) - \left(\frac{1}{4} \cancel{-1+1}\right)\right)$$
$$+ \left(\left(2500 - 1000 + 100\right) - \left(\cancel{4-8+4}\right)\right)$$

$$= 1600 + 1/2 = 1600.5$$

7. Approximate using Simpson's rule with $n = 4$:

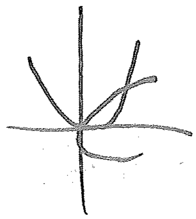
$$\int_0^1 x^2 \mathrm{d}x$$

$$\frac{b-a}{3n}\left(f(x_0)+4f(x_1)+2f(x_2)+\cdots+4f(x_{n-1})+f(x_n)\right)$$

$$\frac{0.25}{3}\left(0^2+4(0.25)^2+2(0.5)^2+4(0.75)^2+1^2\right)$$

$$= 1/3$$

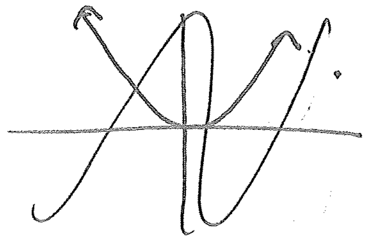
8. Find the area enclosed by $y = x^2$ and $x = y^2$.

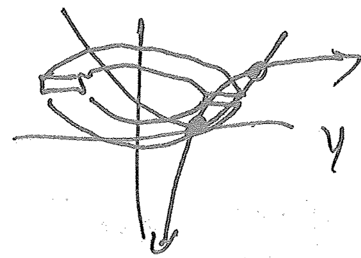


$$\int_0^1 \left(\sqrt{x} - x^2\right) dx$$

$$= \frac{2}{3}x^{3/2} - \frac{x^3}{3}\Big]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

9. Find the volume of the solid formed by rotating the region bounded by $y = \log_2 x$ and $y = (x-1)^2$ about the y-axis.





$y = 1$

$y = 0$

$x = 2^y$

$x = \sqrt{y} + 1$

Annuli

$$V_i = \pi\left((\sqrt{y}+1)^2 - (2^y)^2\right) dy$$

$$V = \int_0^1 \pi\left((\sqrt{y}+1)^2 - 4^y\right) dy$$

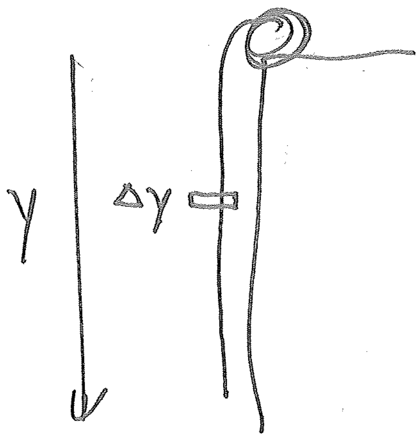
$$= \pi \int_0^1 (y + 2\sqrt{y} + 1 - 4^y)\, dy$$

$$= \pi\left[\frac{y^2}{2} + \frac{4}{3}y^{3/2} + y - \frac{4^y}{\ln 4}\right]_0^1$$

$$= \pi\left(\left(\frac{1}{2} + \frac{4}{3} + 1 - \frac{4}{\ln 4}\right) - \left(\frac{-1}{\ln 4}\right)\right)$$

$$= \pi\left(\frac{17}{6} - \frac{3}{\ln 4}\right) \approx 2.103$$

10. Suppose a chain 500ft in length weighs 3000lbs. Find the work required to lift the chain to the top of a building 500ft high.



$$F_i = \frac{3000}{500} \Delta y = 6 \Delta y$$

$$W_i = 6y \Delta y$$

$$W = \int_0^{500} 6y \, dy = 3y^2 \Big]_0^{500} = 750{,}000 \text{ ft-lb}$$

11. (a) Give the Maclaurin Series for $f(x) = \sin x$

(b) Use the alternating series test to determine how many terms are needed to approximate $\sin \frac{\pi}{12}$ accurate to 6 decimals.

(c) Find the indicated approximation

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

(b) $\sin \frac{\pi}{12} = \sum_{n=0}^{\infty} (-1)^n \frac{(\pi/12)^{2n+1}}{(2n+1)!}$ b_n

n	b_{n+1}
1	$1.02 \cdot 10^{-5}$
2	$1.7 \cdot 10^{-8}$ ✓

$n = 2$

(c) $\frac{\pi}{12} - \frac{(\pi/12)^3}{3!} + \frac{(\pi/12)^5}{5!} = 0.258819$

12. (a) Find the Taylor series expansion of $f(x) = \sin x$ at $a = \pi$

(b) Use Taylor's inequality ($R_n(x) \leq \frac{M}{(n+1)!}|x-a|^{n+1}$, where $\left|f^{(n+1)}(x)\right| \leq M$) to determine how many terms of the series are needed to approximate $\sin\frac{11\pi}{12}$ accurate to 6 decimals.

(c) Find the indicated approximation.

(a)

$f(x) = \sin x$ $\quad$ $f(\pi) = 0$

$f'(x) = \cos x$ $\quad$ $f'(\pi) = -1$

$f''(x) = -\sin x$ $\quad$ $f''(\pi) = 0$

$f'''(x) = -\cos x$ $\quad$ $f'''(\pi) = 1$

$f^{(4)}(x) = \sin x$ $\quad$ $f^{(4)}(\pi) = 0$

$$\sum_{n=0}^{\infty} (-1)^{n-1} \frac{\cancel{x^{2n+1}}}{(2n+1)!} (x-\pi)^{2n+1}$$

(b) $M = 1$

n	$\frac{1}{(n+1)!}\left(\frac{\pi}{12}\right)^{n+1}$
1	0.03
2	0.003
3	$1.96 \cdot 10^{-4}$
4	$1.02 \cdot 10^{-5}$
5	$4.47 \cdot 10^{-7}$

$n = 5$

(c) $\cancel{-\frac{11\pi}{12} + \frac{(11\pi/12)^3}{3!} - \frac{(11\pi/12)^5}{5!}}$

$\cancel{= 0.258819}$

$$-(-\pi/12) + \frac{(-\pi/12)^3}{3!} - \frac{(-\pi/12)^5}{5!} = 0.258819$$