Name: Key #2

Each problem is worth the indicated number of points. Show all your work for full credit; numerical or graphical estimates are unacceptable unless specifically requested. Use of calculators with symbolic algebraic capability such as the TI-89 is prohibited and will result in a grade of zero on this exam.

1. (15 pts) Determine whether the sequence converges or diverges. If it converges, find the limit.

   (a) $\{\cos \pi n\}$

   (b) $\{\sin \pi n\}$

   (c) $\{\frac{2^n}{n^2}\}$

   (d) $\left\{\frac{(-1)^n}{n^2}\right\}$

(a) $\{1, -1, 1, -1, \ldots\}$ oscillates $\Rightarrow$ dvgs

(b) $\{0, 0, \cdots\}$ $\lim_{n\to\infty} \sin \pi n = 0$

(c) $\lim_{n\to\infty} \frac{2^n}{n^2} = \lim_{n\to\infty} \frac{(\ln 2) 2^n}{2n} = \lim_{n\to\infty} \frac{(\ln 2)^2 2^n}{2}$

$= \infty$ dvgs

(d) $\lim_{n\to\infty} \left|\frac{(-1)^n}{n^2}\right| = \lim_{n\to\infty} \frac{1}{n^2} = 0$

$\Rightarrow \left\{\frac{(-1)^n}{n^2}\right\}$ cnvgs to 0 by the

sqz Thm.

2. (20 pts) For the function $f(x) = \sin x$

(a) Give the generic form of a Maclaurin series

(b) Find the Maclaurin series for $f(x)$

(c) Use the Maclaurin series to approximate $\sin \frac{\pi}{12}$, accurate to 5 decimals.

(a) $$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

(b) $$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

(c)

n	$b_{n+1} = \left(\frac{\pi}{12}\right)^{2n+3} / (2n+3)!$
1	$1.0 \cdot 10^{-3}$
2	$1.7 \cdot 10^{-8}$ ✓

$$\sin \frac{\pi}{12} \approx \frac{\pi}{12} - \frac{(\pi/12)^3}{3!} + \frac{(\pi/12)^5}{5!}$$

$$= 0.25882$$

3. (20 pts) Show that, accurate to 5 decimal places,

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

$$\int_N^\infty \frac{1}{x^6}dx = \left.\frac{x^{-5}}{-5}\right]_N^\infty = \frac{-1}{5\infty^5} + \frac{1}{5N^5} = \frac{1}{5N^5}$$

$$\frac{1}{5N^5} < 0.000005$$

$$N > \sqrt[5]{\frac{1}{5(0.000005)}} = 8.3 \quad \text{Use } N = 9$$

$$\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \frac{1}{6^6} + \frac{1}{7^6} + \frac{1}{8^6} + \frac{1}{9^6}$$

$$= 1.01734 \approx \pi^6/945$$

4. (20 pts)

(a) Find a power series representation of $f(x) = \frac{1}{1-x^2}$

(b) Use part 4a to find a power series representation of $g(x) = \frac{x^2}{(1-x^2)^2}$

(c) Find the radii of convergence of the series in parts 4a and 4b.

(a) $\sum_{n=0}^{\infty} x^{2n}$ $\qquad \frac{d}{dx}\frac{1}{1-x^2} = \frac{2x}{(1-x^2)^2}$

(b) $g(x) = \frac{x}{2} f'(x) = \frac{x}{2}\frac{d}{dx}\sum_{n=0}^{\infty} x^{2n}$

$= \frac{x}{2}\sum_{n=0}^{\infty} 2nx^{2n-1} = \sum_{n=0}^{\infty} nx^{2n}$

(c) (4a) $|x^2| < 1 \Leftrightarrow -1 < x < 1 \Rightarrow R = 1$

(4b) $\lim_{n\to\infty}\left|\frac{(n+1)x^{2(n+1)}}{nx^{2n}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}x^2\right|$

$= |x^2| < 1 \Leftrightarrow -1 < x < 1 \Leftrightarrow R = 1$

5. (25 pts) Determine whether the series converges. If it converges, find the sum.

(a) $\sum_{n=3}^{\infty} \frac{4}{n^2-4}$

(b) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$

(d) $\sum_{n=1}^{\infty} \frac{1}{n+\ln n}$

(e) $\sum_{n=1}^{\infty} \frac{(-1)^n \cos \pi n}{n}$

(a) $\frac{A}{n-2} + \frac{B}{n+2} = \frac{4}{n^2-4}$

$A(n+2) + B(n-2) = 4$

$(A+B)n + 2A - 2B = 4$

$A = -B$

$2A + 2A = 4$

$A = 1 \quad B = -1$

$$\sum_{n=3}^{\infty}\left(\frac{1}{n-2} - \frac{1}{n+2}\right)$$

$$S_n = \left(1 - \cancel{\tfrac{1}{5}}\right) + \left(\tfrac{1}{2} - \cancel{\tfrac{1}{6}}\right) + \left(\tfrac{1}{3} - \cancel{\tfrac{1}{7}}\right) + \left(\tfrac{1}{4} - \cancel{\tfrac{1}{8}}\right) + \left(\cancel{\tfrac{1}{5}} - \ldots\right)$$

$$+ \cdots + \left(\cancel{\tfrac{1}{n-6}} - \cancel{\tfrac{1}{n-2}}\right) + \left(\cancel{\tfrac{1}{n-5}} - \tfrac{1}{n-1}\right) + \left(\cancel{\tfrac{1}{n-4}} - \tfrac{1}{n}\right)$$

$$+ \left(\cancel{\tfrac{1}{n-3}} - \tfrac{1}{n+1}\right) + \left(\cancel{\tfrac{1}{n-2}} - \tfrac{1}{n+2}\right)$$

$$= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} - \frac{1}{n-1} - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2}$$

$$\lim_{n\to\infty} S_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} = \tfrac{25}{12}$$

(b) $\ln n \leq n \Rightarrow \sum \frac{1}{\ln n} \geq \sum \frac{1}{n}$ dvgt p-series $\Rightarrow \sum \frac{1}{\ln n}$ dvgt by comp test

(c) $\sum \left|\frac{(-1)^n}{3^n}\right| = \sum \frac{1}{3^n}$ cnvg geom. $\Rightarrow \sum \frac{(-1)^n}{3^n}$ (absly) cnvgt

$$-\frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} = -\frac{1}{3}\sum_{n=1}^{\infty}(1/9)^{n-1} + \frac{1}{9}\sum_{n=1}^{\infty}(1/9)^{n-1}$$

$$= -\frac{2}{9}\sum_{n=1}^{\infty}(1/9)^{n-1} = -2/9\left(\frac{1}{1-1/9}\right) = -2/9\left(\frac{1}{8/9}\right) = -1/4$$

6. (Bonus 10 pts) A function $f(x)$ is defined by

$$f(x) = 1 + 2x + 3x^2 + x^3 + 2x^4 + 3x^5 + x^6 + 2x^7 + 3x^8 + \dots$$

Find the interval of convergence of the series and find an explicit formula for $f(x)$.

$$= (1 + x^3 + x^6 + x^9) + 2x(1 + x^3 + x^6 + x^9 + \cdots) + 3x^2(1 + x^3 + x^6 + x^9 + \cdots)$$

$$= (1 + 2x + 3x^2) \underbrace{\sum_{n=1}^{\infty} (x^3)^{n-1}}_{} = (1 + 2x + 3x^2) \frac{1}{1 - x^3}$$

$$= \frac{1 + 2x + 3x^2}{1 - x^3}$$ geom ; cngt for $|x^3| < 1 \Leftrightarrow -1 < x < 1 \Leftrightarrow$

$R = 1$

(d) $\lim_{n\to\infty} \dfrac{1/n}{\frac{1}{n+\ln n}} = \lim_{n\to\infty} \dfrac{n+\ln n}{n} \overset{\infty/\infty}{=} \lim_{n\to\infty} \dfrac{1+1/n}{1} = 1 \Rightarrow$ by limit

comp ~~test~~ $\sum \frac{1}{n+\ln n}$ dvgt since $\sum \frac{1}{n}$ dvgt p-series

(e) $\sum \frac{(-1)^n \cos \pi n}{n} = \sum \frac{1}{n}$ dvgt p-series.