Name: KEY #4

Each problem is worth 20 points. Show all your work for full credit; numerical or graphical estimates are unacceptable unless specifically requested. Use of calculators with symbolic algebraic capability such as the TI-89 is prohibited and will result in a grade of zero on this exam.

Work all of problems 1-3, at least one from 4-5, and at least one from 6-9. You may work one additional problem as a bonus problem worth 10 points; if you work more than one bonus problem I will only grade the first one. Make sure you indicate which problem is to be graded as a bonus; if you fail to do so I will grade the first optional problem you complete out of 10.

1. For the function $f(x) = x^2$ on $[0, 2]$.

   (a) Find the average value f_{ave} of $f(x)$.

   (b) Find a number c such that $f(c) = f_{\text{ave}}$.

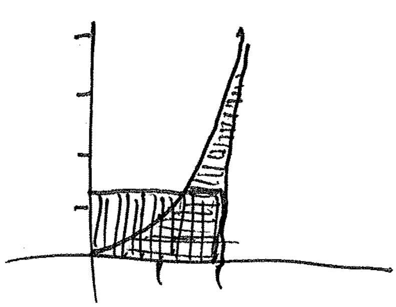
   (c) Illustrate the significance of the number c with a graph that equates areas.

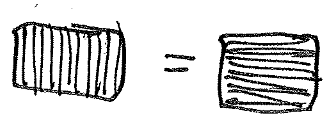
(a) $\frac{1}{2}\int_0^2 x^2 dx = \frac{x^3}{6}\Big]_0^2 = \frac{8}{6} = \frac{4}{3}$

(b) $c^2 = f(c) = 4/3$

$c = 2\sqrt{3}/3 \approx 1.15$

(c)





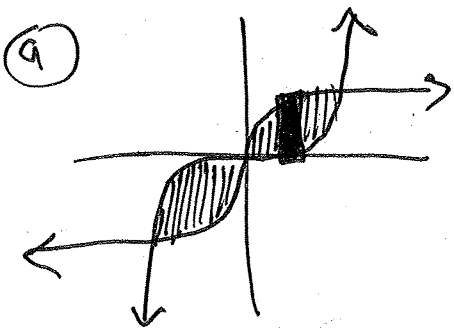
2. For the functions $f(x) = \sqrt[3]{x}$ and $g(x) = x^3$:

   (a) Sketch the area enclosed by the graphs of $f(x)$ and $g(x)$. Draw a typical slice of the area.

   (b) Write an expression for the area of the i^{th} slice.

   (c) Find the total area by integrating your expression from the previous step.

(a)



Using Symmetry

(b) $\left(\sqrt[3]{x_i} - x_i^3\right)\Delta x$

(c) $2\int_0^1 \left(\sqrt[3]{x} - x^3\right)dx = 2\left[\frac{3}{4}x^{4/3} - \frac{1}{4}x^4\right]_0^1$

$= 2\left(\frac{3}{4} - \frac{1}{4}\right) = 1$

3. Find the length of the parametric curve given by

$$\begin{cases} x = \ln(\sin t) + \ln(\cos t) \\ y = 2t \end{cases}$$

for $0.1 \leq t \leq 0.5$.

$$\int_{0.1}^{0.5} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\left(\frac{dx}{dt}\right)^2 = (\cot t - \tan t)^2$$

$$= \cot^2 t - 2 + \tan^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = 2^2 = 4$$

$$= \int_{0.1}^{0.5} \sqrt{\cot^2 t + 2 + \tan^2 t}\, dt = \int_{0.1}^{0.5} \sqrt{(\cot t + \tan t)^2}\, dt$$

$$= \int_{0.1}^{0.5} (\cot t + \tan t)\, dt = \ln \sin t - \ln \cos t \Big]_{0.1}^{0.5}$$

$$\approx 1.69$$

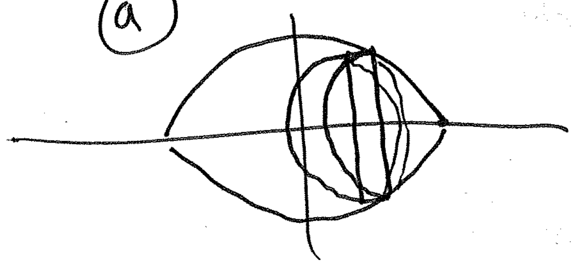
4. For the region bounded by $y = \cos\frac{\pi x}{2}$ and the x-axis: $-1 \le x \le 1$

(a) Sketch the solid formed by rotating the region about the x-axis. Include a typical slice of the volume.

(b) Find the volume of one slice.

(c) Find the total volume by integrating your expression from the previous step.

(a)



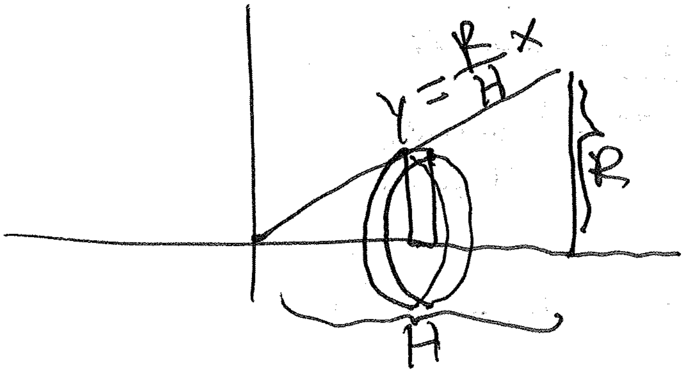
(b) $\pi \cos^2\left(\frac{\pi x_i}{2}\right) \Delta x$

(c) $$\pi \int_{-1}^{1} \cos^2\left(\frac{\pi x}{2}\right) dx$$

$$= \pi \int_{-1}^{1} \frac{1+\cos \pi x}{2}\, dx = 2\pi \left[\frac{x}{2} + \frac{1}{2\pi} \sin \pi x\right]_0^1$$

$$= 2\pi\left(\left(\frac{1}{2}+0\right) - (0+0)\right) = \pi$$

5. Prove that the volume of a cone with height H and base radius R is given by $V = \frac{1}{3}\pi R^2 H$.

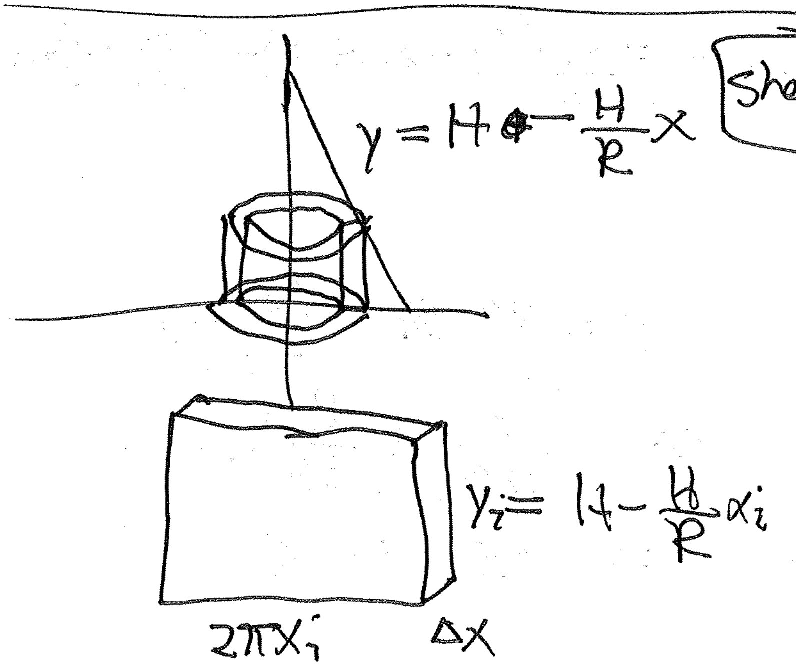


$\boxed{\text{discs}}$

$$V = \int_0^H \pi\left(\frac{R}{H}x\right)^2 dx = \pi\int_0^H \frac{R^2}{H^2}x^2\,dx$$

$$= \pi\left[\frac{R^2}{H^2}\frac{x^3}{3}\right]_0^H = \pi\frac{R^2 H^3}{3H^2} = \frac{1}{3}\pi R^2 H$$

$\boxed{\text{shells}}$



$$V = \int_0^R 2\pi x\left(H - \frac{H}{R}x\right)dx$$

$$= 2\pi\int_0^R \left(Hx - \frac{H}{R}x^2\right)dx$$

$$= 2\pi\left[\frac{Hx^2}{2} - \frac{Hx^3}{3R}\right]_0^R$$

$$= 2\pi\left(\frac{HR^2}{2} - \frac{HR^3}{3R}\right)$$

$$= \pi\left(HR^2 - \frac{2}{3}HR^2\right)$$

$$= \frac{1}{3}\pi R^2 H$$

$$V_i = 2\pi x_i\left(H - \frac{H}{R}x_i\right)\Delta x$$

6. Suppose the work required to stretch a spring from 8" to 10" is 15ft – lb and the work required to stretch it from 10" to 12" is 25ft – lb. Find the spring's natural length and its spring constant.

$$15 = \int_{\cancel{c-4/6}\;\, 4/6-c}^{\cancel{c-5/6}\;\, 5/6-c} kx\,dx = \frac{kx^2}{2}\Bigg]_{\cancel{c-4/6}\;\, 4/6-c}^{5/6-c\;\, \cancel{c-5/6}}$$

$$= \frac{k}{2}\left(\overset{(5/6-c)^2}{\cancel{(c-5/6)}} - (4/6-c)^2\right)$$

$$= \frac{k}{2}\left(\left(\frac{25}{36} - \frac{5}{3}c + c^2\right) - \left(\frac{16}{36} - \frac{4}{3}c + c^2\right)\right)$$

$$= \frac{k}{2}\left(\frac{9}{36} - \frac{c}{3}\right)$$

$$25 = \int_{5/6-c}^{1-c} kx\,dx = \frac{kx^2}{2}\Bigg]_{5/6-c}^{1-c}$$

$$= \frac{k}{2}\left((1-c)^2 + (5/6-c)^2\right) = \frac{k}{2}\left(\left(1 - 2c + c^2\right) - \left(\frac{25}{36} - \frac{5}{3}c + \right.\right.$$

$$= \frac{k}{2}\left(\frac{11}{36} - \frac{c}{3}\right)$$

$$15 = \frac{k}{2}\left(\frac{9}{36} - \frac{c}{3}\right)$$

$$25 = \frac{k}{2}\left(\frac{11}{36} - \frac{c}{3}\right)$$

$$10 = \frac{k}{2}\left(\frac{2}{36}\right) \qquad k = 360$$

$$25 = 180\left(\frac{11}{36} - \frac{c}{3}\right)$$

$$c = \left(\frac{-25}{180} + \frac{11}{36}\right)3$$

$$= 0.5' = 6''.$$

7. Find the centroid of the lamina bounded by $f(x) = -x^2 + 16$ and the x-axis.

$$A = \int_{-4}^{4} (-x^2 + 16)\,dx = 2\left[\frac{-x^3}{3} + 16x\right]_0^4$$

$$= 2\left(-64/3 + 64\right) = \frac{256}{3}$$

$$0 = \bar{x} = \frac{1}{A}\int_{-4}^{4} (-x^3 + 16x)\,dx = \frac{3}{256}\left[\frac{-x^4}{4} + 8x^2\right]_{-4}^{4}$$

$$= 0$$

$$\bar{y} = \frac{1}{A}\int_{-4}^{4} \frac{1}{2} f(x)^2 = \frac{3}{512}\int_{-4}^{4} (x^4 - 32x^2 + 256)\,dx$$

$$= \frac{3}{256}\int_{0}^{4} (x^4 - 32x^2 + 256)\,dx = \frac{3}{256}\left[\frac{x^5}{5} - \frac{32x^3}{3} + 256x\right]_0^4$$

$$= 6.4$$

For problems 8-9, you should:

(a) Draw a picture that divides the problem into infinitesimal segments

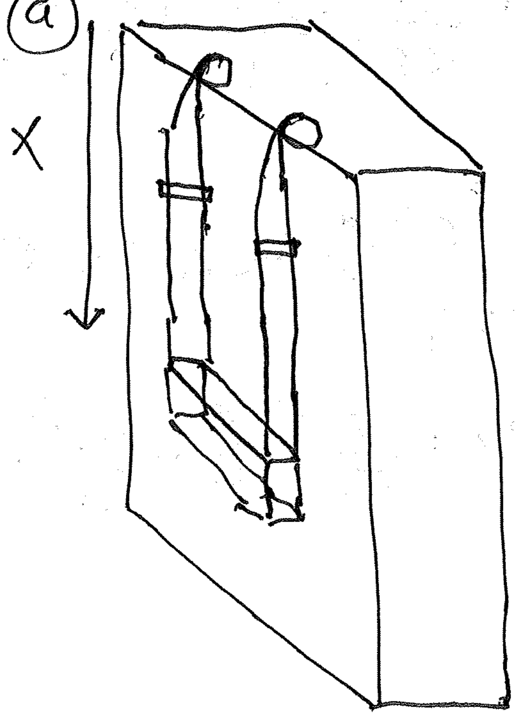
(b) Solve the problem for one segment

(c) Form an integral using your answer from the preceding step

(d) Evaluate the integral

8. Suppose a painter's scaffold is held suspended from the top of a building 400m high by four cables just long enough to reach the ground and weighing 150kg each. The total weight of the scaffold, painters and equipment is 300kg. Find the work required to bring the scaffold from ground level to half-way up the building.

(a)



$$W_{scaffold} = 300(9.8)(200) = 588{,}000\,J = 588\,kJ$$

$$W_{lower} = 4\left(\tfrac{1}{2}\right)(150)(200)(9.8) = 588\,kJ$$

(c) $$W_{upper} = \int_0^{200} 29.4x\,dx$$

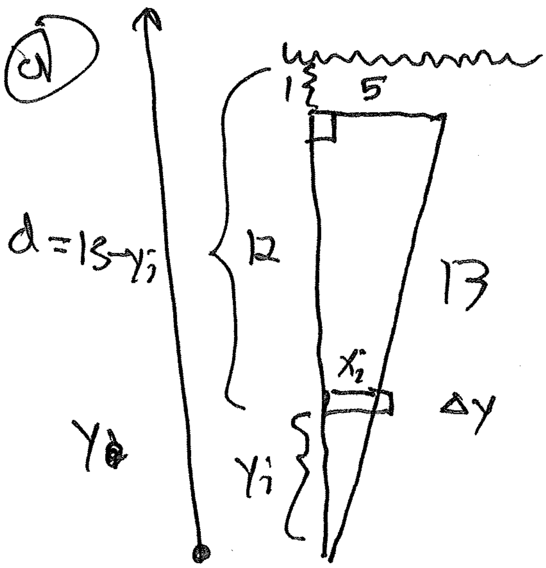
(d) $$= 14.7x^2\Big]_0^{200} = 588\,kJ$$

(b) $$W_i = 4\left(\frac{150}{200}\right)\Delta x\,(9.8)\,x_i = 29.4x\,dx$$

$$W_{tot} = 1764\,kJ$$

9. Suppose a plate has the shape of a right triangle with side lengths 5m, 12m, 13m is submerged vertically underwater with the pointy end down and the opposite end (the 5m side) parallel to and 1m below the water's surface. Find the hydrostatic force on one side of the plate.

a



$$\frac{5}{12} = \frac{x_i}{y_i}$$

$$x_i = \frac{5}{12} y_i$$

$$P_i = 1000(9.8)(13 - y_i)$$

$$A_i = x_i \Delta y = \frac{5}{12} y_i \Delta y$$

b $F_i = P_i A_i = 9800(13 - y_i)\frac{5}{12} y_i \, dy$

c $F = \int_0^{12} 9800(13 - y)\frac{5}{12} y \, dy$

d $= \frac{12,250}{3}\int_0^{12} (13y - y^2)\, dy$

$$= \frac{12,250}{3}\left[\frac{13y^2}{2} - \frac{y^3}{3}\right]_0^{12}$$

$$= 1,470,000 \text{ N}$$