You should work through this as you would an actual exam. Time yourself, and try to solve each problem alone without using your notes or your book.

1. Find the following indefinite integrals:

   (a) $\int xe^x \mathrm{d}x$

$$\int \underbrace{x}_{u} \underbrace{e^x \mathrm{d}x}_{dv} = xe^x - \int e^x \mathrm{d}x$$
$$= xe^x - e^x + C$$

   (b) $\int x^3 \sin \pi x \mathrm{d}x$

$$\int \underbrace{x^3}_{u} \underbrace{\sin \pi x}_{dv} = -\frac{x^3}{\pi}\cos \pi x - \int -\frac{3x^2}{\pi}\cos \pi x \mathrm{d}x$$
$$= -\frac{x^3}{\pi}\cos \pi x + \int \underbrace{\frac{3x^2}{\pi}}_{u} \underbrace{\cos \pi x \mathrm{d}x}_{dv}$$
$$= -\frac{x^3}{\pi}\cos \pi x + \left(\frac{3x^2}{\pi^2}\sin \pi x - \int \underbrace{\frac{6x}{\pi^2}}_{u} \underbrace{\sin \pi x \,\mathrm{d}x}_{dv}\right)$$
$$= -\frac{x^3}{\pi}\cos \pi x + \frac{3x^2}{\pi^2}\sin \pi x - \left(-\frac{6x}{\pi^3}\cos \pi x - \int -\frac{6}{\pi^3}\cos \pi x \mathrm{d}x\right)$$
$$= -\frac{x^3}{\pi}\cos \pi x + \frac{3x^2}{\pi^2}\sin \pi x + \frac{6x}{\pi^3}\cos \pi x - \int \frac{6}{\pi^3}\cos \pi x \mathrm{d}x$$
$$= -\frac{x^3}{\pi}\cos \pi x + \frac{3x^2}{\pi^2}\sin \pi x + \frac{6x}{\pi^3}\cos \pi x - \frac{6}{\pi^4}\sin \pi x \mathrm{d}x$$

(c) $\int e^{4x}\cos 5x\mathrm{d}x$

$$\begin{aligned}
\int \underbrace{e^{4x}}_{dv}\underbrace{\cos 5x}_{u}\,\mathrm{d}x &= \frac{1}{4}\cos 5xe^{4x} - \int -\frac{5}{4}e^{4x}\sin 5x\mathrm{d}x \\
&= \frac{1}{4}\cos 5xe^{4x} + \frac{5}{4}\int \underbrace{e^{4x}}_{dv}\underbrace{\sin 5x}_{u}\,\mathrm{d}x \\
&= \frac{1}{4}\cos 5xe^{4x} + \frac{5}{4}\left(\frac{1}{4}\sin 5xe^{4x} - \int \frac{5}{4}\cos 5xe^{4x}\mathrm{d}x\right) \\
&= \frac{1}{4}\cos 5xe^{4x} + \frac{5}{16}\sin 5xe^{4x} - \frac{25}{16}\int \cos 5xe^{4x}\mathrm{dx} \Longrightarrow \\
\frac{41}{16}\int e^{4x}\cos 5x &= \frac{1}{4}\cos 5xe^{4x} + \frac{5}{16}\sin 5xe^{4x} + C \Longrightarrow \\
\int e^{4x}\cos 5x &= \frac{4}{41}\cos 5xe^{4x} + \frac{5}{41}\sin 5xe^{4x} + C
\end{aligned}$$

(d) $\int \sqrt{4+4x^2}\mathrm{d}x$

$$\begin{aligned}
x &= \tan\theta \\
\mathrm{d}x &= \sec^2\theta \mathrm{d}\theta \\
\theta &= \tan^{-1} x \\
\int \sqrt{4+4x^2}\mathrm{d}x &= \int \sqrt{4+4\tan^2\theta}\sec^2\theta\mathrm{d}\theta \\
&= 2\int \sec^3\theta\mathrm{d}\theta \text{(ignore the 2 for now)} \\
\int \sec^3\theta\mathrm{d}\theta &= \underbrace{\int \sec^2\theta}_{dv} \underbrace{\sec\theta}_{u} \mathrm{d}\theta \\
&= \sec\theta\tan\theta - \int \tan\theta\sec\theta\tan\theta\mathrm{d}\theta \\
&= \sec\theta\tan\theta - \int \tan^2\theta\sec\theta\mathrm{d}\theta \\
&= \sec\theta\tan\theta - \int \left(\sec^2\theta - 1\right)\sec\theta\mathrm{d}\theta \\
&= \sec\theta\tan\theta - \int \sec^3\theta\mathrm{d}\theta + \int \sec\theta\mathrm{d}\theta \Longrightarrow \\
2\int \sec^3\theta\mathrm{d}\theta &= \sec\theta\tan\theta + \int \sec\theta\mathrm{d}\theta \\
&= \sec\theta\tan\theta + \ln|\sec\theta+\tan\theta| + C \\
&= \sec\tan^{-1}x\tan\left(\tan^{-1}x\right) + \ln\left|\sec\left(\tan^{-1}x\right) + \tan\left(\tan^{-1}x\right)\right| + C \\
&= \sqrt{1+\tan^2\left(\tan^{-1}x\right)}\tan\left(\tan^{-1}x\right) + \ln\left|\sqrt{1+\tan^2\left(\tan^{-1}x\right)} + \tan\left(\tan^{-1}x\right.\right. \\
&= x\sqrt{1+x^2} + \ln\sqrt{1+x^2} + x + C
\end{aligned}$$

(e) $\int \frac{x^3}{\sqrt{x^2-1}}\mathrm{d}x$

$$\begin{aligned}
x &= \sec\theta \\
\mathrm{d}x &= \sec\theta\tan\theta\mathrm{d}\theta \\
\theta &= \sec^{-1}x \\
\int \frac{x^3}{\sqrt{x^2-1}}\mathrm{d}x &= \int \frac{\sec^3\theta}{\sqrt{\sec^2\theta-1}}\sec\theta\tan\theta\mathrm{d}\theta \\
&= \int \frac{\sec^3\theta}{\tan\theta}\sec\theta\tan\theta\mathrm{d}\theta \\
&= \int \sec^4\theta\mathrm{d}\theta \\
&= \int \left(1+\tan^2\theta\right)\sec^2\theta\mathrm{d}\theta \\
u &= \tan\theta \\
\mathrm{d}u &= \sec^2\theta\mathrm{d}\theta \\
\int \left(1+\tan^2\theta\right)\sec^2\theta\mathrm{d}\theta &= \int \left(1+u^2\right)\mathrm{d}u \\
&= u+\frac{u^3}{3}+C \\
&= \tan\theta+\frac{\tan^3\theta}{3}+C \\
&= \tan\left(\sec^{-1}x\right)+\frac{\tan^3\left(\sec^{-1}x\right)}{3}+C \\
&= \sqrt{\sec^2\left(\sec^{-1}x\right)-1}+\frac{\left(\sec^2\left(\sec^{-1}x\right)-1\right)^{3/2}}{3}+C \\
&= \sqrt{x^2-1}+\frac{\left(x^2-1\right)^{3/2}}{3}+C
\end{aligned}$$

(f) $\int \sin^2\theta\mathrm{d}\theta$

$$\begin{aligned}
\int \sin^2\theta\mathrm{d}\theta &= \int \frac{1-\cos 2\theta}{2}\mathrm{d}\theta \\
&= \int \left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta \\
&= \frac{\theta}{2}-\frac{1}{4}\sin 2\theta+C
\end{aligned}$$

(g) $\int \cos^9 \theta \mathrm{d}\theta$

$$\begin{aligned}
\int \cos^9 \theta \mathrm{d}\theta &= \int \left(\cos^2 \theta\right)^4 \cos\theta \mathrm{d}\theta \\
&= \int \left(1 - \sin^2 \theta\right)^4 \cos\theta \mathrm{d}\theta \\
u &= \sin\theta \\
\mathrm{d}u &= \cos\theta \mathrm{d}\theta \\
&= \int \left(1 - u^2\right)^4 \mathrm{d}u \\
&= \int \left(u^8 - 4u^6 + 6u^4 - 4u^2 + 1\right) \mathrm{d}u \\
&= \frac{u^9}{9} - \frac{4u^7}{7} + \frac{6u^5}{5} - \frac{4u^3}{3} + u + C \\
&= \frac{\cos^9 \theta}{9} - \frac{4\cos^7 \theta}{7} + \frac{6\cos^5 \theta}{5} - \frac{4\cos^3 \theta}{3} + \cos\theta + C
\end{aligned}$$

(h) $\int \frac{x^3-1}{x^3+1}\mathrm{d}x$

$$\begin{aligned}
\int \frac{x^3-1}{x^3+1}\mathrm{d}x &= \int \left(1-\frac{1}{x^3+1}\right)\mathrm{d}x \\
\frac{1}{x^3+1} &= \frac{1}{(x+1)(x^2-x+1)} \\
&= \frac{A}{x+1}+\frac{Bx+C}{x^2-x+1} \Longrightarrow \\
1 &= A(x^2-x+1)+(Bx+C)(x+1) \Longrightarrow \\
1 &= (A+B)x^2+(-A+B+C)x+(A+C) \Longrightarrow \\
-A=B \quad &, \quad C=1-A \Longrightarrow \\
0 &= -A+B+C=-2A+1-A=-3A+1 \Longrightarrow \\
A &= 1/3 \Longrightarrow \\
B=-1/3 \quad &, \quad C=2/3 \Longrightarrow \\
\int \frac{x^3-1}{x^3+1} &= \int \left(1-\left(\frac{1/3}{x+1}+\frac{-1/3x+2/3}{x^2-x+1}\right)\right)\mathrm{d}x \\
&= \int \mathrm{d}x - \frac{1}{3}\int \frac{1}{x+1}\mathrm{d}x + \frac{1}{3}\int \frac{x-2}{x^2-x+1}\mathrm{d}x \\
&= x-\ln|x+1|+\frac{1}{6}\int \frac{2x-1}{x^2-x+1}\mathrm{d}x - \frac{1}{6}\int \frac{3}{x^2-x+1}\mathrm{d}x \\
&= x-\ln|x+1|+\frac{1}{6}\ln\left|x^2-x+1\right| - \underbrace{\frac{1}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}x}_{(*)}
\end{aligned}$$

$$\begin{aligned}
u &= x-\frac{1}{2} \\
\mathrm{d}u &= \mathrm{d}x \\
(*) &= \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u \\
&= \frac{1}{2}\frac{1}{\sqrt{3/4}}\tan^{-1}\frac{u}{\sqrt{3/4}}+C \\
&= \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{2\sqrt{3}}{3}\left(x-\frac{1}{2}\right)\right)+C \Longrightarrow \\
\int \frac{x^3-1}{x^3+1} &= x-\ln|x+1|+\frac{1}{6}\ln\left|x^2-x+1\right| - \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{2\sqrt{3}}{3}\left(x-\frac{1}{2}\right)\right)+C
\end{aligned}$$

(i) $\int \frac{2x^4+x^3+3x^2+1}{x^6+2x^4+x^2}\mathrm{d}x$

$$\begin{aligned}\int \frac{2x^4+x^3+3x^2+1}{x^6+2x^4+x^2}\mathrm{d}x &= \int\left(\frac{1}{x^2}+\frac{1}{x^2+1}+\frac{x}{\left(x^2+1\right)^2}\right)\mathrm{d}x \\ &= -\frac{1}{x}+\tan^{-1}x+\frac{1}{2}\int\frac{2x}{\left(x^2+1\right)^2}\mathrm{d}x \\ &= -\frac{1}{x}+\tan^{-1}x+\frac{1}{2}\ln\left|x^2+1\right|+C\end{aligned}$$

2. Use Simpson's Rule with $n = 8$ to approximate

$$\int_0^\pi \sin x \mathrm{d}x$$

$$\begin{aligned}&\frac{\pi/8}{3}\left(\sin 0+4\sin\frac{\pi}{8}+2\sin\frac{2\pi}{8}+4\sin\frac{3\pi}{8}+2\sin\frac{4\pi}{8}+4\sin\frac{5\pi}{8}+2\sin\frac{6\pi}{8}+4\sin\frac{7\pi}{8}+\sin\frac{8\pi}{8}\right)\\ \approx\ & 2.00026917\end{aligned}$$

3. Find the following definite integrals:

(a) $\int_{-1}^{1}\frac{1}{x^{2/3}}\mathrm{d}x$

$$\begin{aligned}\int_{-1}^{1}\frac{1}{x^{2/3}}\mathrm{d}x &= \int_{-1}^{0}\frac{1}{x^{2/3}}\mathrm{d}x+\int_{0}^{1}\frac{1}{x^{2/3}}\mathrm{d}x \\ &= \lim_{t\to0^-}\int_{-1}^{t}\frac{1}{x^{2/3}}\mathrm{d}x+\lim_{t\to0^+}\int_{t}^{1}\frac{1}{x^{2/3}}\mathrm{d}x \\ &= \lim_{t\to0^-}\int_{-1}^{t}x^{-2/3}\mathrm{d}x+\lim_{t\to0^+}\int_{t}^{1}x^{-2/3}\mathrm{d}x \\ &= \lim_{t\to0^-}\left[3x^{1/3}\right]_{-1}^{t}+\lim_{t\to0^+}\left[3x^{1/3}\right]_{t}^{1} \\ &= \lim_{t\to0^-}\left(3t^{1/3}-3(-1)^{1/3}\right)+\lim_{t\to0^+}\left(3-3t^{1/3}\right) \\ &= 6\end{aligned}$$

(b) $\int_1^\infty \frac{3x^2-2x-3}{x^4+3x^3+x^2+3x}\mathrm{d}x$ Start by using partial fractions to get:

$$\begin{aligned}
\int_1^\infty \frac{3x^2-2x-3}{x^4+3x^3+x^2+3x}\mathrm{d}x &= \int_1^\infty \left(\frac{2x}{x^2+1}-\frac{1}{x+3}-\frac{1}{x}\right)\mathrm{d}x \\
&= \lim_{t\to\infty}\int_1^t \left(\frac{2x}{x^2+1}-\frac{1}{x+3}-\frac{1}{x}\right)\mathrm{d}x \\
&= \lim_{t\to\infty}\left[\ln\left|x^2+1\right|-\ln|x+3|-\ln|x|\right]_1^t \\
&= \lim_{t\to\infty}\left[\ln\left|\frac{x^2+1}{x(x+3)}\right|\right]_1^t \\
&= \lim_{t\to\infty}\left(\ln\left|\frac{t^2+1}{t(t+3)}\right|-\ln\left|\frac{1}{2}\right|\right) \\
&= \lim_{t\to\infty}\left(\ln\left|\frac{1+1/t^2}{1+3/t}\right|-\ln\frac{1}{2}\right) \\
&= -\ln\frac{1}{2}
\end{aligned}$$

4. Find the area inside the curve given by $r=\sin 6\theta$ The entire pattern is traced out on $[0,2\pi]$. So

$$\begin{aligned}
A &= \frac{1}{2}\int_0^{2\pi}\sin^2 6\theta\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{2\pi}\frac{1-\cos 12\theta}{2}\mathrm{d}\theta \\
&= \frac{1}{4}\int_0^{2\pi}(1-\cos 12\theta)\,\mathrm{d}\theta \\
&= \frac{1}{4}\left[\theta-\frac{1}{12}\sin 12\theta\right]_0^{2\pi} \\
&= \frac{1}{4}\left((2\pi-0)-(0-0)\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

5. [Redacted]

6. Prove the reduction formula:

$$\int \cos^n x\mathrm{d}x = \frac{1}{n}\cos^{n-1}x\sin x+\frac{n-1}{n}\int\cos^{n-2}x\mathrm{d}x$$

$$
\begin{aligned}
\int \cos^n x \mathrm{d}x &= \int \underbrace{\cos^{n-1} x}_{u} \underbrace{\cos x \mathrm{d}x}_{dv} \\
&= \sin x \cos^{n-1} x - \int \sin x (n-1) \cos^{n-2} x (-\sin x) \mathrm{d}x \\
&= \sin x \cos^{n-1} x + (n-1) \int \sin^2 x \cos^{n-2} x \mathrm{d}x \\
&= \sin x \cos^{n-1} x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \mathrm{d}x \\
&= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \mathrm{d}x + (n-1) \int \cos^n x \mathrm{d}x \Longrightarrow \\
n \int \cos^n x \mathrm{d}x &= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \mathrm{d}x \Longrightarrow \\
\int \cos^n x \mathrm{d}x &= \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x \mathrm{d}x
\end{aligned}
$$

7. Use integrals to prove that the area of a circle of radius r is πr^2.

$$
\begin{aligned}
A &= 4 \int_0^r \sqrt{r^2 - x^2} \mathrm{d}x \\
x &= r \sin \theta \\
\mathrm{d}x &= r \cos \theta \mathrm{d}\theta \\
\theta &= \sin^{-1} \tfrac{x}{r} \\
A &= 4 \int_{\theta(0)}^{\theta(r)} \sqrt{r^2 - \sin^2 \theta} r \cos \theta \mathrm{d}\theta \\
&= 4 \int_{\sin^{-1} 0}^{\sin^{-1} 1} \sqrt{r^2 \cos^2 \theta} r \cos \theta \mathrm{d}\theta \\
&= 4r^2 \int_0^{\pi/2} \cos^2 \theta \mathrm{d}\theta \\
&= 4r^2 \int_0^{\pi/2} \frac{1 + \cos 2\theta}{2} \mathrm{d}\theta \\
&= 4r^2 \left[\frac{\theta}{2} + \frac{1}{4} \sin 2\theta \right]_0^{\pi/2} \\
&= 4r^2 \left(\left(\frac{\pi}{4} + \frac{1}{4} \sin \pi \right) - \left(0 + \frac{1}{4} \sin 0 \right) \right) \\
&= 4r^2 \left(\frac{\pi}{4} \right) \\
&= \pi r^2
\end{aligned}
$$

8. Determine whether the integral is convergent:

$$\int_2^\infty \frac{1}{(\ln x)^2} \mathrm{d}x$$

$$\begin{aligned}
x &\geq \ln x \Longrightarrow \\
x \ln x &\geq (\ln x)^2 \Longrightarrow \\
\frac{1}{x \ln x} &\leq \frac{1}{(\ln x)^2} \Longrightarrow \\
\int_2^\infty \frac{1}{(\ln x)^2} \mathrm{d}x &\geq \int_2^\infty \frac{1}{x \ln x} \mathrm{d}x \\
&= \lim_{t \to \infty} \int_2^t \frac{1}{x \ln x} \mathrm{d}x \\
(u = \ln x, \mathrm{d}u = \frac{1}{x} \mathrm{d}x) &= \lim_{t \to \infty} \int_{u(2)}^{u(t)} \frac{\mathrm{d}u}{u} \\
&= \lim_{t \to \infty} [\ln u]_{\ln 2}^{\ln t} \\
&= \lim_{t \to \infty} (\ln \ln t - \ln \ln 2) \\
&= \ln \ln \infty - \ln \ln 2 \\
&= \infty
\end{aligned}$$

So both integrals diverge.