Name: KEY

Each problem is worth the indicated number of points. Show all your work for full credit; numerical or graphical estimates are unacceptable unless specifically requested. You may work one bonus problem; if you work more than one I will only grade the first one.

1. (20 pts) Find the indefinite integral:

$$\int \frac{x^4+x^3+4x^2+3x+4}{x^5+4x^3+4x} dx = \frac{A}{x} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}$$

$$x^4+x^3+4x^2+3x+4 = A(x^2+2)^2 + (Bx+C)x(x^2+2) + (Dx+E)x$$

$$= \cancel{Ax^4} + \cancel{4Ax^2} + 4A + \cancel{Bx^4} + \cancel{2Bx^2} + \cancel{Cx^3} + \cancel{2Cx} + \cancel{Dx^2} + \cancel{Ex}$$

$$= (A+B)x^4 + Cx^3 + (4A+2B+D)x^2 + (2C+E)x + 4A$$

$A+B=1$

$C=1$

$4A+2B+D=4$

$2C+E=3$

$4A=4$

$A=1$

$B=0$

$C=1$

$D=0$

$E=1$

$$\int\left(\frac{1}{x} + \frac{1}{x^2+2} + \frac{1}{(x^2+2)^2}\right)dx$$

$$= \ln|x| + \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} \cancel{- \frac{1}{2(x^2+2)}} + C$$

$$\int \frac{1}{(x^2+2)^2}dx = \frac{-1}{4}\int \frac{1}{x}\frac{-4/x^3}{(1+2/x^2)^2}dx$$

u, du

$$= -\frac{1}{4}\left(\frac{1}{x}\cdot\frac{-1}{(1+2/x^2)} - \int\frac{-1}{(1+2/x^2)}\cdot\frac{-1}{x^2}dx\right)$$

$$= \frac{x}{4(x^2+2)} + \frac{1}{4}\int \frac{1}{x^2+2}dx = \frac{x}{4(x^2+2)} + \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

$$= \ln|x| + \frac{5\sqrt{2}}{8}\tan^{-1}\left(\frac{\sqrt{2}}{2}x\right) + \frac{x}{4(x^2+2)} + C$$

3. (15 pts) Find the indefinite integral:

$$\int \sin x \cos x dx$$

(a) Using integration by parts

(b) By using a double angle formula

Use trigonometric identities to check that your answers agree.

(a) $\int \underbrace{\sin x}_{u} \underbrace{\cos x\, dx}_{dv} = \sin^2 x - \int \cos x \sin x \, dx$

$2\int \sin x \cos x dx = \sin^2 x + C$

$\int \sin x \cos x dx = \frac{1}{2}\sin^2 x + C$

(b) $\int \sin x \cos x dx = \frac{1}{2}\int 2\sin x \cos x dx$

$= \frac{1}{2}\int \sin(2x) dx = -\frac{1}{4}\cos(2x) + C$ Use $c = 1/4$

$\frac{1}{4}\cos(2x) =$

$\frac{1}{2}\sin^2 x = \frac{1}{4}(1-\cos 2x)$

Use $C = -\frac{1}{2}$. Then $\frac{1}{2}\sin^2 x + C =$

$-\frac{1}{2} + \frac{1}{2}\sin^2 x = -\frac{1}{2}\cos^2 x$

$= -\frac{1}{2}\left(\frac{1+\cos 2x}{2}\right)$

$= -\frac{1}{4} - \frac{1}{4}\cos 2x$

$$\sec \tan^{-1} 8 = \sqrt{\tan^2 \tan^{-1} 8 + 1}$$
$$= \sqrt{65}$$

5. (15 pts) Find the definite integral:

$$\int_0^8 \sqrt{1+x^2}\,dx$$

$x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$\theta = \tan^{-1} x$

$$\int_{\tan^{-1}0}^{\tan^{-1}8} \underbrace{\sqrt{1+\tan^2\theta}}_{u = \sec\theta} \; \underbrace{\sec^2\theta \, d\theta}_{dv}$$

$$= \int_0^{\tan^{-1}8} \sec^3\theta \, d\theta$$

$$= \Big[\sec\theta \tan\theta\Big]_0^{\tan^{-1}8} - \int_0^{\tan^{-1}8} \tan\theta \sec\theta \tan\theta \, d\theta$$

$$= \Big[\sec\theta \tan\theta\Big]_0^{\tan^{-1}8} - \int_0^{\tan^{-1}8} (\sec^2\theta - 1)\sec\theta \, d\theta$$

$$\int_0^{\tan^{-1}8} \sec^3\theta \, d\theta = \Big[\sec\theta \tan\theta\Big]_0^{\tan^{-1}8} - \int_0^{\tan^{-1}8} \sec^3\theta \, d\theta + \int_0^{\tan^{-1}8} \sec\theta \, d\theta$$

$$\int_0^{\tan^{-1}8} \sec^3\theta \, d\theta = \frac{1}{2}\Big[\sec\theta \tan\theta\Big]_0^{\tan^{-1}8} + \frac{1}{2}\Big[\ln|\sec\theta + \tan\theta|\Big]_0^{\tan^{-1}8}$$

$$= 4\sqrt{65} + \frac{1}{2}\ln|\sqrt{65}+8|$$

7. (15 pts) Find the area inside the curve given in polar coordinates by $r = \sin(2\theta)$

$$\frac{1}{2}\int_0^{2\pi} \sin^2(2\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1-\cos(4\theta)}{2}\, d\theta$$

$$= \frac{1}{4}\int_0^{2\pi} (1-\cos(4\theta))\, d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{1}{4}\sin(4\theta)\right]_0^{2\pi}$$

$$= \frac{1}{4}(2\pi - 0) = \frac{\pi}{2}$$

9. (Bonus 10 pts) Determine whether the integral is convergent:

$$\int_{100}^{\infty} \frac{1}{\ln(\ln(\ln(x)))} dx$$

for $x \geq 100$,

$\ln\ln\ln x \leq \ln\ln x \leq \ln x \leq x$.

So $\dfrac{1}{\ln\ln\ln x} \geq \dfrac{1}{x}$.

But $\displaystyle\int_{100}^{\infty} \frac{1}{x} dx = \lim_{t\to\infty} \ln|x| \Big]_{100}^{t}$

$= \displaystyle\lim_{t\to\infty} \ln t - \ln 100 = \infty$.

So both integrals diverge.