Exam 2

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Name: Key

Each problem is worth the indicated number of points. Show all your work for full credit; numerical or graphical estimates are unacceptable unless specifically requested. You may work one bonus problem; if you work more than one I will only grade the first one.

1. (20 pts) Find the indefinite integral:

$$\int \frac{4x^3 - 6x^2 + x - 2}{x^2\left(x^2+1\right)} \mathrm{d}x$$

$x^2(x^2+1)\left(\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}\right) = 4x^3 - 6x^2 + x - 2$

$Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2 =$ "

$(A+C)x^3 + (B+D)x^2 + Ax + B =$ "

$B=-2$

$A=1$

$C=3$

$D=-4$

$\int\left(\frac{1}{x} - \frac{-2}{x^2} + \frac{3x-4}{(x^2+1)}\right)dx$

$= \ln|x| + \frac{2}{x} + \frac{3}{2}\int \frac{2x}{x^2+1}dx - 4\int\frac{1}{x^2+1}d$

$= \ln|x| + \frac{2}{x} + \frac{3}{2}\ln|x^2+1| - 4\tan^{-1}x + C$

2. (10 pts) Find the definite integral:

$$\int_1^\infty \frac{1}{x^2} dx$$

$$= \lim_{t \to \infty} \int_1^t \frac{1}{x^2} dx$$

$$= \lim_{t \to \infty} \left(-\frac{1}{x} \right]_1^t$$

$$= \lim_{t \to \infty} -\frac{1}{t} + 1 = 1$$

3. (10 pts) Find the indefinite integral:

$$\int \underbrace{e^{2x}}_{dv} \underbrace{\sin x}_{u} dx$$

$$= \frac{1}{2}\sin x\, e^{2x} - \int \frac{1}{2} \underbrace{e^{2x}}_{dv} \underbrace{\cos x}_{u}\, dx$$

$$= \frac{1}{2}\sin x\, e^{2x} - \frac{1}{2}\left(\frac{1}{2}\cos x\, e^{2x} - \int \frac{1}{2} e^{2x}(-\sin x)\right)$$

$$= \frac{1}{2}\sin x\, e^{2x} - \frac{1}{4}\cos x\, e^{2x} + \frac{1}{4}\int e^{2x}\sin x$$

$$\frac{5}{4}\int e^{2x}\sin x\, dx = \frac{1}{2}\sin x\, e^{2x} - \frac{1}{4}\cos x\, e^{2x} + C$$

$$\int e^{2x}\sin x\, dx = \frac{2}{5}\sin x\, e^{2x} - \frac{1}{5}\cos x\, e^{2x} + C$$

4. (15 pts) Find the indefinite integral:

$$\int \csc^{10}\theta d\theta$$

$$= \int (1+\cot^2\theta)^4 \csc^2\theta \, d\theta$$

$u = \cot\theta \quad du = -\csc^2\theta \, d\theta$

$$= -\int (1+u^2)^4 \, du$$

$$= -\int (u^8 + 4u^6 + 6u^4 + 4u^2 + 1) \, du$$

$$= -\left(\frac{\cot^9\theta}{9} + \frac{4\cot^7\theta}{7} + \frac{6\cot^5\theta}{5} + \frac{4\cot^3\theta}{3} + \cot\theta \right) + C$$

5. (15 pts) Find the definite integral:

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx$$

$x = \sin\theta$

$dx = \cos\theta \, d\theta$

$\theta = \sin^{-1} x$

$$= \int_{\theta(-1)}^{\theta(1)} \sqrt{1-\sin^2\theta}\ \cos\theta \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= \frac{\theta}{2} + \frac{1}{4}\sin 2\theta \Big]_{-\pi/2}^{\pi/2}$$

$$= \left(\frac{\pi}{4} + 0\right) - \left(-\frac{\pi}{4} + 0\right) = \frac{\pi}{2}$$

6. (15 pts) Use Simpson's Rule with $n = 8$ to approximate

$$\int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}x$$

$$\frac{0.25}{3}\Big(\sqrt{1-1^2} + 4\sqrt{1-(-0.75)^2} + 2\sqrt{1-(-0.5)^2}$$

$$+ 4\sqrt{1-(-0.25)^2} + 2\sqrt{1-0^2} + 4\sqrt{1-0.25^2}$$

$$+ 2\sqrt{1-0.5^2} + 4\sqrt{1-0.75^2} + \sqrt{1-1^2}\Big)$$

$$= 1.5418$$

7. (15 pts) Find the area inside the curve given in polar coordinates by $r = 1 + \cos\theta$

$$A = \frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (\cos^2\theta + 2\cos\theta + 1)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2} + 2\cos\theta + 1\right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{3}{4} + \frac{1}{4}\cos 2\theta + \cos\theta\right) d\theta$$

$$= \frac{3\theta}{4} + \frac{1}{8}\sin 2\theta + \sin\theta \Big]_0^{2\pi}$$

$$= \left(\frac{6\pi}{4} + 0 + 0\right) - (0 + 0 + 0)$$

$$= 3\pi/2$$

8. (Bonus 10 pts) Prove the reduction formula:

$$\int \csc^n x dx = -\frac{1}{n-1} \cot x \csc^{n-2} x + \frac{n-2}{n-1} \int \csc^{n-2} x dx$$

$$\int \csc^n x\, dx = \int \underbrace{\csc^{n-2} x}_{u} \underbrace{\csc^2 x\, dx}_{dv}$$

$$= -\csc^{n-2} x \cot x - \int -\cot x (n-2) \csc^{n-3} x \, (-\csc x \cot x) dx$$

$$= -\csc^{n-2} x \cot x - (n-2) \int \cot^2 x \cdot \csc^{n-2} x dx$$

$$= -\csc^{n-2} x \cot x - (n-2) \int (\csc^2 x - 1) \csc^{n-2} x dx$$

$$= -\csc^{n-2} x \cot x - (n-2) \int \csc^n x dx + (n-2) \int \csc^{n-2} x$$

$$(n-1) \int \csc^n x dx = -\csc^{n-2} x \cot x + (n-2) \int \csc^{n-2} x$$

$$\int \csc^n x\, dx = \frac{-1}{n-1} \csc^{n-2} x \cot x + \frac{n-2}{n-1} \int \csc^{n-2} x dx$$

9. (Bonus 10 pts) Determine whether the integral is convergent:

$$\int_1^\infty \frac{x}{e^{2x^2}+4}\,\mathrm{d}x$$

$$\frac{x}{e^{2x^2}+4} \leq \frac{2xe^{x^2}}{e^{2x^2}+4}$$

And $\int_1^\infty \frac{2xe^{x^2}}{e^{2x^2}+4}\,dx$

$$= \lim_{t\to\infty} \int_1^t \frac{2xe^{x^2}}{e^{2x^2}+4}\,dx \qquad u = e^{x^2}, \quad du = 2xe^{x^2}\,dx \qquad \frac{du}{u^2+4}$$

$$= \lim_{t\to\infty} \frac{1}{2}\tan^{-1}\left(\frac{e^{x^2}}{2}\right)\Big]_1^t$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(e/2)\right) < \infty$$

So both integrals converge.

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