You should work through this as you would an actual exam. Time yourself, and try to solve each problem alone without using your notes or your book.

1. Suppose a motorist causes his car to crash on a dry day. His car weighs 1000 kg and leaves skid marks 170 m long. If the coefficient of friction of rubber on dry asphalt is 0.75 and the speed limit is 55 mph (≈ 24.59m/s), show that he was speeding when the crash occurred. (Use the fact that the absolute value of the force due to friction equals the downward force on the car times the coefficient of friction, and that force equals mass times acceleration).

   $v(0) = v_0 = ?, s(0) = s_0 = 0, s(b) = 170, v(b) = 0$

$$
\begin{aligned}
a(t) &= -1000(9.8)(0.75)/1000 \\
&= -7.35 \\
v(t) &= -7.35t + v_0 \\
s(t) &= -3.675t^2 + v_0 t + s_0 \\
&= -3.675t^2 + v_0 t \\
0 &= v(b) \\
&= -7.35b + v_0 \\
v_0 &= 7.35b \\
170 &= s(b) \\
&= -3.675b^2 + v_0 b \\
&= -3.675b^2 + 7.35b^2 \\
&= 3.675b^2 \\
b^2 &= \frac{170}{3.675} \\
b &= \sqrt{\frac{170}{3.675}} \\
v_0 &= 7.35\sqrt{\frac{170}{3.675}} \\
&\approx 49.99 \\
&> 24.59
\end{aligned}
$$

2. Approximate

$$\int_0^5 \left[x^3 - 3x^2\right] \mathrm{d}x$$

using a right-hand sum with n subintervals, where

(a) $n = 10$

$a = 0, b = 5, \Delta x = 0.5, x_0 = 0, x_1 = 0.5, x_2 = 1, x_3 = 1.5, x_4 = 2, x_5 = 2.5, x_6 = 3, x_7 = 3.5, x_8 = 4, x_9 = 4.5, x_10 = 5$

$$\begin{aligned}\Delta x \sum_{k=1}^{10} \left[x_k^3 - 3x_k^2\right] &= 0.5(\quad 0.5^3 - 3(0.5)^2 + 1^3 - 3(1)^2 + 1.5^3 - 3(1.5)^2 + 2^3 - 3(2)^2 + \\ &\quad 2.5^3 - 3(2.5)^2 + 3^3 - 3(3)^2 + 3.5^3 - 3(3.5)^2 + 4^3 - 3(4)^2 + \\ &\quad 4.5^3 - 3(4.5)^2 + 5^3 - 3(5)^2) \\ &= \quad 44.6875\end{aligned}$$

(b) $n = 20$

37.734375

(c) $n = 100$

32.509375

(d) $n = 999$

31.37521906

Write out the complete sum for part 2a, including the values of $a, b, \Delta x$ and all the x_k's. Just use your calculator for parts 2b-2d.

3. Use the definition of the integral (NOT FTC) to evaluate

$$\int_0^5 \left[x^3 - 3x^2\right] \mathrm{d}x$$

Draw a diagram interpreting this integral in terms of areas.

$$
\begin{aligned}
\int_0^5 \left[x^3 - 3x^2\right] \mathrm{d}x &= \lim_{n\to\infty} \Delta x \sum_{k=1}^{n} \left[x_k^3 - 3x_k^2\right] \\
&= \lim_{n\to\infty} \frac{5}{n} \sum_{k=1}^{n} \left[\left(\frac{5k}{n}\right)^3 - 3\left(\frac{5k}{n}\right)^2\right] \\
&= \lim_{n\to\infty} \frac{5}{n} \left[\frac{125}{n^3} \sum_{k=1}^{n} k^3 - \frac{75}{n^2} \sum_{k=1}^{n} k^2\right] \\
&= \lim_{n\to\infty} \left[\frac{625}{n^4} \frac{n^2 (n+1)^2}{4} - \frac{375}{n^3} \frac{n(n+1)(2n+1)}{6}\right] \\
&= \lim_{n\to\infty} \left[\frac{625}{4} \left(\frac{n}{n}\right)^2 \left(\frac{n+1}{n}\right)^2 - \frac{375}{6} \frac{n}{n} \frac{n+1}{n} \frac{2n+1}{n}\right] \\
&= \lim_{n\to\infty} \left[\frac{625}{4} (1)^2 \left(1 + \frac{1}{n}\right)^2 - \frac{375}{6} (1) \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right)\right] \\
&= \frac{625}{4} - \frac{375}{6}(2) \\
&= \frac{125}{4}
\end{aligned}
$$

4. Use appropriate area formulas to find

$$\int_{-3}^{7} f(x) \mathrm{d}x$$

where

$$f(x) = \begin{cases} 4\sqrt{1 - \frac{x^2}{9}} & \text{if} \quad x \leq 0 \\ -\frac{4}{7}x + 4 & \text{if} \quad x > 0 \end{cases}$$

From -3 to 0, $f(x)$ is a quarter-ellipse with axes 3 and 4, so the area there is $\frac{1}{4}\pi(3)(4) = 3\pi$. From 4 to 7, $f(x)$ is a triangle, so the area there is $\frac{1}{2}(7)(4) = 14$. So the total area is $14 + 3\pi$

5. Given that $\int_4^5 f(x)\mathrm{d}x = 7$ and $\int_4^5 g(x)\mathrm{d}x = 3$, find

$$\int_4^5 \left[2f(x) - 3g(x)\right] \mathrm{d}x$$

$$
\begin{aligned}
2\int_4^5 f(x)\mathrm{d}x - 3\int_4^5 g(x)\mathrm{d}x &= 2(7) - 3(3) \\
&= 5
\end{aligned}
$$

6. Find the following indefinite integrals:

(a)

$$\int e^x \mathrm{d}x = e^x + C$$

(b)

$$\int 3^x \mathrm{d}x = \frac{3^x}{\ln 3} + C$$

(c)

$$\int \frac{\ln x}{x} \mathrm{d}x$$

$$\begin{aligned} u &= \ln x \\ \mathrm{d}u &= \frac{1}{x}\mathrm{d}x \\ \int \frac{\ln x}{x}\mathrm{d}x &= \int u\mathrm{d}u \\ &= \frac{u^2}{2} + C \\ &= \frac{(\ln x)^2}{2} + C \end{aligned}$$

(d)

$$\begin{aligned} \int \frac{x^4 - 3x^2 + \sqrt{x^3} - 5x - \sqrt{x} + 2}{x^2}\mathrm{d}x &= \int \left[x^2 - 3 + x^{-1/2} - \frac{5}{x} - x^{-3/2} + 2x^{-2}\right]\mathrm{d}x \\ &= \frac{x^3}{3} - 3x + 2\sqrt{x} - 5\ln|x| + 2\sqrt{x} - \frac{2}{x} + C \end{aligned}$$

(e)

$$\int \sin t \mathrm{d}t = -\cos t + C$$

(f)

$$\int \sec^2 \theta \mathrm{d}\theta = \tan t + C$$

(g)

$$\int -\frac{1}{x\sqrt{x^2 - 1}} = \csc^{-1} x + C$$

(h)

$$\int \sin^3 t \mathrm{d}t$$

$$\begin{aligned}
\int \sin^3 t \mathrm{d}t &= \int \sin^2 t \sin t \mathrm{d}t \\
&= \int \left(1-\cos^2 t\right) \sin t \mathrm{d}t \\
u &= \cos t \\
\mathrm{d}u &= -\sin t \mathrm{d}t \\
\int \left(1-\cos^2 t\right) \sin t \mathrm{d}t &= -\int \left(1-\cos^2 t\right)(-\sin t)\,\mathrm{d}t \\
&= -\int \left(1-u^2\right) \mathrm{d}u \\
&= -\left(u-\frac{u^3}{3}\right)+C \\
&= -u+\frac{u^3}{3}+C \\
&= -\cos t+\frac{\cos^3 t}{3}+C
\end{aligned}$$

7. Evaluate the following definite integrals:

(a)

$$\begin{aligned}
\int_8^{27} \frac{1}{\sqrt[3]{x}} \mathrm{d}x &= \int_8^{27} x^{-1/3} \mathrm{d}x \\
&= \left.\frac{x^{2/3}}{2/3}\right]_8^{27} \\
&= \left.\frac{3}{2} x^{2/3}\right]_8^{27} \\
&= \frac{3}{2}\left(27^{2/3}-8^{2/3}\right) \\
&= \frac{3}{2}(9-4) \\
&= \frac{15}{2}
\end{aligned}$$

(b)

$$\begin{aligned}
\int_2^{10} \frac{5t+3}{t^2+1}\mathrm{d}t &= \int_2^{10} \frac{5t}{t^2+1}\mathrm{d}t + \int_2^{10} \frac{3}{t^2+1}\mathrm{d}t \\
&= \frac{5}{2}\int_2^{10} \frac{2t}{t^2+1}\mathrm{d}t + 3\int_2^{10} \frac{1}{t^2+1}\mathrm{d}t \\
&= \frac{5}{2}\int_{u(2)}^{u(10)} \frac{1}{u}\mathrm{d}u + 3\int_2^{10} \frac{1}{t^2+1}\mathrm{d}t (u = t^2+1, \mathrm{d}u = 2t\mathrm{d}t \\
&= \frac{5}{2}\left[\ln|u|\right]_5^{101} + 3\left[\tan^{-1} t\right] 2^{10} \\
&= \frac{5}{2}\left(\ln 101 - \ln 5\right) + 3\left(\tan^{-1} 10 - \tan^{-1} 2\right) \\
&= \frac{5}{2}\ln\frac{101}{5} + 3\left(\tan^{-1} 10 - \tan^{-1} 2\right)
\end{aligned}$$

(c)

$$\begin{aligned}
\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{6}{y^2+1}\mathrm{d}y &= 6\tan^{-1} y\big]_{\sqrt{3}/3}^{\sqrt{3}} \\
&= 6\left(\tan^{-1}\sqrt{3} - \tan^{-1}\frac{\sqrt{3}}{3}\right) \\
&= 6\left(\frac{\pi}{3} - \frac{\pi}{6}\right) \\
&= 6\frac{\pi}{6} \\
&= \pi
\end{aligned}$$

(d)

$$\begin{aligned}
\int_4^9 \sqrt{3t^7 - 5t^4}\mathrm{d}t &= \int_4^9 \sqrt{t^4\left(3t^3-5\right)}\mathrm{d}t \\
&= \int_4^9 t^2\sqrt{3t^3-5}\mathrm{d}t \\
&= \frac{1}{9}\int_4^9 \sqrt{3t^3-5}9t^2\mathrm{d}t \\
&= \frac{1}{9}\int_{u(4)}^{u(9)} \sqrt{u}\mathrm{d}u (u = 3t^3-5, \mathrm{d}u = 9t^2\mathrm{d}t) \\
&= \frac{1}{9}\frac{u^{3/2}}{3/2}\Bigg]_{187}^{2182} \\
&= \frac{2}{27}\left(2182^{3/2} - 187^{3/2}\right)
\end{aligned}$$

8. Suppose the velocity of a particle at time t is given by

$$v(t) = t^2 - 10t + 16$$

Find

(a) The displacement of the particle at time $t = 10$ (assume the particle starts at the origin).

$$\begin{aligned}
\int_0^{10} v(t)\mathrm{d}t &= \int_0^{10} \left[t^2 - 10t + 16\right]\mathrm{d}t \\
&= \frac{t^3}{3} - 5t^2 + 16t\Bigg]_0^{10} \\
&= \left(\frac{10^3}{3} - 5(10)^2 + 16(10)\right) - 0 \\
&= -6.67
\end{aligned}$$

(b) The distance traveled by the particle for $0 \le t \le 10$.

$$\begin{aligned}
\int_0^{10} |v(t)|\,\mathrm{d}t &= \int_0^{10} \left|t^2 - 10t + 16\right| \mathrm{d}t \\
&= \int_0^{2} \left(t^2 - 10t + 16\right) \mathrm{d}t + \int_2^{8} \left(-t^2 + 10t - 16\right) \mathrm{d}t + \int_8^{10} \left(t^2 - 10t + 16\right) \mathrm{d}t \\
&= \left[\frac{t^3}{3} - 5t^2 + 16t\right]_0^2 + \left[-\frac{t^3}{3} + 5t^2 - 16t\right]_2^8 + \left[\frac{t^3}{3} - 5t^2 + 16t\right]_8^{10} \\
&= \left(\frac{8}{3} - 20 + 32\right) - 0 + \\
&\quad \left(-\frac{512}{3} + 320 - 128\right) - \left(-\frac{8}{3} + 20 - 32\right) + \\
&\quad \left(\frac{1000}{3} - 500 + 160\right) - \left(\frac{512}{3} - 320 + 128\right) \\
&= \frac{196}{3} \\
&= 65.\bar{3}
\end{aligned}$$

9. Suppose the total population of spectacled flying foxes in Queensland has been decreasing by $r(t) = 2010(0.99)^t$ animals/yr, where t is the number of years since 1900. Find

$$\int_0^{109} r(t)\mathrm{d}t$$

$$\begin{aligned}
\int_0^{109} r(t)\mathrm{d}t &= \int_0^{109} 2010(0.99)^t \mathrm{d}t \\
&= 2010 \frac{(0.99)^t}{\ln 0.99}\bigg]_0^{109} \\
&= \frac{2010}{\ln 0.99}\left(0.99^{109} - 0.99^0\right) \\
&= 19,122.9 \\
&\approx 19,123
\end{aligned}$$

Give units and interpret your answer. In the 109 years since 1900 (i.e. since 1900), we have lost 19,123 bats.

10. Find

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \ln \cot^{-1} \sqrt{t^2 + 5}\mathrm{d}t = \ln \cot^{-1} \sqrt{x^2 + 5}$$