Exam 1

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Name: [illegible]

Each problem is worth 15 points. Show all your work for full credit; numerical or graphical estimates are unacceptable unless specifically requested.

1. Use the definition of the definite integral to find

$$\int_3^6 x^3 \mathrm{d}x$$

$$= \lim_{n\to\infty} \Delta x \sum_{k=1}^{n} x_k^3 = \lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} \left(3+\frac{3k}{n}\right)^3$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} \left(27 + \frac{81k}{n} + \frac{81k^2}{n^2} + \frac{27k^3}{n^3}\right)$$

$$= \lim_{n\to\infty} \left[\frac{3}{n}\left(27 \sum_{k=1}^{n} 1 + \frac{81}{n}\sum_{k=1}^{n} k + \frac{81}{n^2}\sum_{k=1}^{n} n^2 + \frac{27}{n^3}\sum_{k=1}^{n} k^3\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{3}{n}\left(27n + \frac{81}{n}\frac{n(n+1)}{2} + \frac{81}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{27}{n^3}\frac{n^2(n+1)^2}{4}\right)\right]$$

$$= \lim_{n\to\infty} \left[81 + \frac{243}{2}\frac{n}{n}\frac{n+1}{n} + \frac{243}{6}\frac{n}{n}\frac{n+1}{n}\frac{2n+1}{n} + \frac{81}{4}\frac{n^2}{n^2}\left(\frac{n+1}{n}\right)^2\right]$$

$$= \lim_{n\to\infty} \left(81 + \frac{243}{2}\left(1+\frac{1}{n}\right) + \frac{243}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + \frac{81}{4}\left(1+\frac{1}{n}\right)^2\right)$$

$$= 81 + 243/2 + 243/3 + 81/4 = 303.75$$

2. Use a midpoint sum with n subintervals to approximate

$$\int_3^6 x^3 \mathrm{d}x$$

for

(a) $n = 6$

(b) $n = \cancel{30}$ 12

(c) ~~$n = 100$~~

(d) ~~$n = 999$~~

Write out the complete sum for part 2a, including the values of $a, b, \Delta x$, and all the x_k's.

(a) $a = 3$

$b = 6$

$\Delta x = \frac{6-3}{6} = 1/2$

$x_0 = 3$

$x_1 = 3.5$

$x_2 = 4$

$x_3 = 4.5$

$x_4 = 5$

$x_5 = 5.5$

$x_6 = 6$

$\frac{1}{2}(3.5^3 + 4^3 + 4.5^3 + 5^3 + 5.5^3 + 6^3) = 352.6875$

(b) ~~313.2675~~

~~(c) 306.591075~~

~~(d) 304.0338447~~

(b) 303.5390625

(a) $\frac{1}{2}(3.25^3 + 3.75^3 + 4.25^3 + 4.75^3 + 5.25^3 + 5.75^3)$

$= 302.90625$

$\bar{x}_1 = 3.25$

$\bar{x}_2 = 3.75$

$\bar{x}_3 = 4.25$

$\bar{x}_4 = 4.75$

$\bar{x}_5 = 5.25$

$\bar{x}_6 = 5.75$

3. Evaluate the following indefinite integrals:

(a) $\int \frac{1}{4^x} dx$

(b) $\int \frac{1}{\sqrt{1-x^2}} dx$

(c) $\int \csc^2 t dt$

(d) $\int \sin\theta d\theta$

(e) $\int \frac{y^2-5y+3}{\sqrt{y}} dy$

(f) $\int \tan x dx$

(a) $= \int \left(\frac{1}{4}\right)^x dx = \frac{1/4^x}{\ln 1/4} + C$

(b) $\sin^{-1} x + C$

(c) $-\cot x + C$

(d) $-\cos\theta + C$

(e) $\int \left(y^{3/2} - 5y^{1/2} + 3y^{-1/2}\right) dy$

$= \frac{y^{5/2}}{5/2} - \frac{5y^{3/2}}{3/2} + \frac{3y^{1/2}}{1/2} + C$

$= \frac{2}{5} y^{5/2} - \frac{10}{3} y^{3/2} + 6y^{1/2} + C$

(f) $= \int \frac{1}{\cos x} \sin x \, dx = -\int \frac{1}{\cos x} (-\sin x \, dx)$

$u = \cos x$

$du = -\sin x \, dx$

$= -\int \frac{1}{u} du = -\ln|u| + C$

$= -\ln|\cos x| + C$

4. Suppose water drips from a leaky drainage pipe onto the floor below at a rate given (in ml/min) by $r(t) = 5 - \frac{5}{t^2+1}$. Find

$$\int_0^{2880} r(t)\mathrm{d}t$$

Give units and interpret your answer.

$$= 5t - 5\tan^{-1} t \Big]_0^{2880}$$

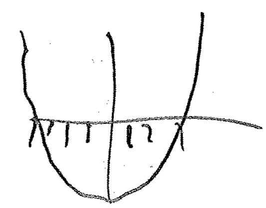
$$= \left(5(2880) - 5\tan^{-1}(2880) \right) - \left(5(0) - 5\tan^{-1} 0 \right)$$

$$\cong 14{,}392$$

in 2880 min (= 2 days), 14,392 ml (= 14.4L) of water has leaked

5. Suppose the velocity of a particle at time t is given (in m/s) by $v(t) = t^2 + t - 12$. Find the distance traveled by the particle during the first 6 seconds.

$$(t+4)(t-3) = 0$$
$$t = 3, -4$$



$$\int_0^6 |v(t)|\,dt = \int_0^3 -v(t)\,dt + \int_3^6 v(t)\,dt$$

$$= \int_0^3 (-t^2 - t + 12)\,dt + \int_3^6 (t^2 + t - 12)\,dt$$

$$= -\frac{t^3}{3} - \frac{t^2}{2} + 12t\Big]_0^3 + \frac{t^3}{3} + \frac{t^2}{2} - 12t\Big]_3^6$$

$$= \left(-\frac{27}{3} - \frac{9}{2} + 36\right) - 0 + \left(\frac{216}{3} + \frac{36}{2} - 72\right) - \left(\frac{27}{3} + \frac{9}{2} - 36\right)$$

$$= 63$$

6. Evaluate:

$$\int_{-3}^{4} \frac{\tan^{-1} t - 1}{t^2 + 1} \mathrm{d}t$$

$$= \int_{-3}^{4} \frac{\tan^{-1} t}{t^2+1} \, dt - \int_{-3}^{4} \frac{1}{t^2+1} \, dt$$

$u = \tan^{-1} t$

$du = \frac{1}{t^2+1} \, dt$

$$= \int_{u(-3)}^{u(4)} u \, du - \int_{-3}^{4} \frac{1}{t^2+1} \, dt$$

$$= \frac{u^2}{2} \Big]_{\tan^{-1} 3}^{\tan^{-1} 4} - \tan^{-1} t \Big]_{-3}^{4}$$

$$= \frac{(\tan^{-1} 4)^2 - (\tan^{-1} 3)^2}{2} - \tan^{-1} 4 + \tan^{-1}(-3$$

≈ -2.476

$u = \tan^{-1} t - 1$

$du = \frac{1}{t^2+1} \, dt$

$$\int_{u(-3)}^{u(4)} u \, du = \frac{u^2}{2} \Big]_{\tan^{-1}(-3)-1}^{\tan^{-1}(4)-1}$$

$$= \frac{(\tan^{-1}(4) - 1)^2 - (\tan^{-1}(-3) - 1)^2}{2}$$

≈ -2.476

7. Evaluate:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{\sqrt{1-t^2}}{\sin^2(t^3)}\mathrm{d}t$$

$$= \frac{\sqrt{1-x^2}}{\sin^2(x^3)}$$

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$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$\cancel{1 + \cot^2\theta = \csc^2\theta}$

$\cancel{t = \sin\theta}$

$t = \sec\theta$

$\cancel{t = \tan\theta}$

$\sqrt{t^2 - 1}$

$\sin^2\theta - 1 = -\cos^2\theta$

t