Final S409

~~Exam 4~~

Name: KEY

Each problem is worth 50 points. Show all your work for full credit; numerical or graphical estimates are unacceptable unless specifically requested.

1. Use the $\epsilon - \delta$ definition of the limit to prove that

$$\lim_{x \to 4} (5x - 3) = 17$$

Let $\varepsilon > 0$, set $\delta = \varepsilon/5$

Then $0 < |x - a| < \delta \Rightarrow$

$-\varepsilon/5 < x - 4 < \varepsilon/5 \Rightarrow$

$-\varepsilon < 5x - 20 < \varepsilon \Rightarrow$

$-\varepsilon + 17 < 5x - 3 < \varepsilon + 17 \Rightarrow$

$-\varepsilon + 17 < y < \varepsilon + 17 \Rightarrow$

$-\varepsilon < y - 17 < \varepsilon \Rightarrow$

$|y - 17| < \varepsilon$

2. Use the definition of the derivative to find

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2+\frac{1}{\sqrt{x}}\right]$$

$$\lim_{h\to 0}\frac{\left((x+h)^2+\frac{1}{\sqrt{x+h}}\right)-\left(x^2+\frac{1}{\sqrt{x}}\right)}{h}$$

$$=\lim_{h\to 0}\left(\frac{(x+h)^2-x^2}{h}+\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{x^2+2xh+h^2-x^2}{h}+\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}\right)$$

$$=\lim_{h\to 0}\left(\frac{2xh+h^2}{h}+\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}\cdot\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\right)$$

$$=\lim_{h\to 0}\left(2x+h+\frac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}\right)$$

$$=\lim_{h\to 0}\left(2x+h \quad \frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}\right)$$

$$=2x-\frac{1}{2x\sqrt{x}}$$

3. Find the derivative:

(a) $\frac{d}{dx}(\ln x)^{(\sin x)}$

(b) $\frac{d}{dx}\frac{x}{\ln x}$

(c) $\frac{d}{dx}\sec x$

(d) $\frac{d}{dx}\sec^{-1} x$

(e) $\frac{d}{dx}\sqrt{x}e^{x}$

(f) $\frac{d}{dt}x^2t^2$

(a) $\ln y = \sin x \ln \ln x$

$\frac{y'}{y} = \cos x \ln \ln x + \sin x \frac{1}{x \ln x}$

$y' = \left(\cos x \ln \ln x + \frac{\sin x}{x \ln x}\right)(\ln x)^{(\sin x)}$

(b) $\frac{\ln x - 1}{(\ln x)^2}$

(c) $\sec x \tan x$

(d) $\frac{1}{|x|\sqrt{x^2-1}}$

(e) $\frac{1}{2\sqrt{x}}e^{x} + \sqrt{x}e^{x}$

(f) $2x^2t$

4. Find

$$\frac{d}{dx} \sin x \cos x$$

in two ways:

(a) By using a trigonometric identity and the chain rule

(b) By using the product rule

Check that your answers agree.

(a) $\frac{d}{dx} \sin x \cos x = \frac{1}{2} \frac{d}{dx} 2 \sin x \cos x$

$= \frac{1}{2} \frac{d}{dx} \sin(2x) = \frac{1}{2}\left(2 \cos(2x)\right) =$

$\cos(2x)$

(b) $\cos^2 x - \sin^2 x = \cos(2x)$

5. State and prove the product rule.

$$\frac{d}{dx} fg = f'g + fg'$$

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h}$$

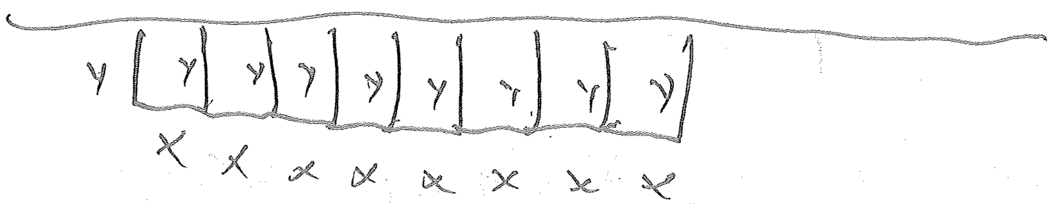
$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \lim_{h\to 0} g(x+h) + f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

($\because$ limit laws)

$$= f'(x)g(x) + f(x)g'(x)$$

($\because$ f, g diffable $\Rightarrow$ g conts)

6. Suppose a farmer wants to build a row of eight cages for his chickens. The back wall will be his barn and the floor will be the ground. The remaining sides of the cages will be constructed from sheets of chicken wire, with the sides of each pair of adjacent cages formed from a single sheet. The top will be open to the sky. The chicken wire costs \$0.75/ft. If each cage must have a footprint of 18ft^2, determine the minimum cost of construction and the dimensions which minimize the cost.



$0.75(8x + 9y) = C$

$xy = 18$

$y = 18/x$

$C(x) = 0.75\left(8x + \frac{162}{x}\right)$

$0 = C' = 0.75\left(8 - \frac{162}{x^2}\right)$

$x^2 = 162/8 = 81/4$

$x = 9/2 = 4.5$ CN

$\lim_{x \to 0^+} C(x) = \infty$

$C(4.5) = \$54$ gmin

$\lim_{x \to \infty} C(x) = \infty$

$x = 4.5$ ft

$y = 18/4.5 = 4$ ft