Final Exam Fa 08

Name: KEY

The optimization problems are worth 50 points; each of the other problems is worth 25 points. Work at least one of the optimization problems and all of the remaining problems; you may work all twelve problems for extra credit. Numerical estimates are unacceptable unless specifically requested; for full credit you must show all your work and use the indicated methods.

1. (25 pts) Use the definition of the limit to show that

$$\lim_{x \to -3} (-5x + 3) = 18$$

Let $\varepsilon > 0$, set $\delta = \varepsilon/5$. Then

$0 < |x-a| < \delta \Rightarrow$

$-\varepsilon/5 < x+3 < \varepsilon/5 \Rightarrow$

$\varepsilon > -5x-15 > -\varepsilon \Rightarrow$

$\varepsilon + 18 > -5x+3 > -\varepsilon + 18 \Rightarrow$

$\varepsilon > (-5x+3) - 18 > -\varepsilon \Rightarrow$

$|(-5x+3) - 18| < \varepsilon$

2. (25 pts) Find

$$\lim_{x \to 0^+} \left(\ln x + \frac{1}{x} \right)$$

$= -\infty + \infty$

$= \lim_{x \to 0^+} \frac{x \ln x + 1}{x}$

$0 \cdot \infty \qquad\qquad -\infty/\infty$

$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2}$

$= \lim_{x \to 0^+} -x = 0$

$= \lim_{x \to 0^+} \frac{x \ln x + 1}{x} = \frac{1}{0^+} = \infty$

3. (25 pts) Find

$$\lim_{t \to 1} \frac{e^{3t} - e^3}{e^{2t} + e^t + 1}$$

$$\lim_{t \to 1} \frac{e^{3t} - e^3}{e^{2t} + e^t + 1} = \frac{e^3 - e^3}{e^2 + e + 1} = 0$$

4. (25 pts) Use the definition of continuity to determine where $f(x)$ is continuous, where

$$f(x) = \begin{cases} x^2+5 & x>4 \\ -3x+33 & x\le 4 \end{cases}$$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} x^2+5 = 16+5 = 21$$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} -3x+33 = -12+33 = 21$$

So $\lim_{x\to 4} f(x) = 21 = -3(4)+33 = f(4)$

and $f(x)$ is cont$\underline{s}$ at 4

also cont$\underline{s}$ on $(-\infty,4)\cup(4,\infty)$

since x^2+5, $-3x+33$ cont$\underline{s}$ on $\mathbb{R}$

Hence $f(x)$ cont$\underline{s}$ on $\mathbb{R}$

5. (25 pts) Use the definition of the derivative to find

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sqrt{x}}$$

$$= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}$$

$$= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}$$

$$= \lim_{h\to 0} \frac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} = \lim_{h\to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}$$

$$= \frac{-1}{2x\sqrt{x}}$$

6. (25 pts) Sketch the graph of $f(x)$ by hand, where

$$f(x) = x^3 - 3x^2 - 45x$$

Show all intercepts, asymptotes, extrema, and inflection points.

No asymptotest

x icepts: $0 = x^3 - 3x^2 - 45x$

$= x(x^2 - 3x - 45)$

$x = 0, \quad \dfrac{3 \pm \sqrt{9 - 4(1)(-45)}}{2}$

y icept: 0

$= 0, \dfrac{3 \pm \sqrt{189}}{2} = \dfrac{3 \pm 3\sqrt{21}}{2}$

$\approx 0, 8.4, -5.4$

$f' = 3x^2 - 6x - 45$

$= 3(x^2 - 2x - 15)$

$= 3(x-5)(x+3)$

$x = -3, 5$

$f'(-10) = + \uparrow$

$f'(0) = - \downarrow$

$f'(10) = + \uparrow$

$f''(x) = 6x - 6$ $\quad x = 1$ poss ip

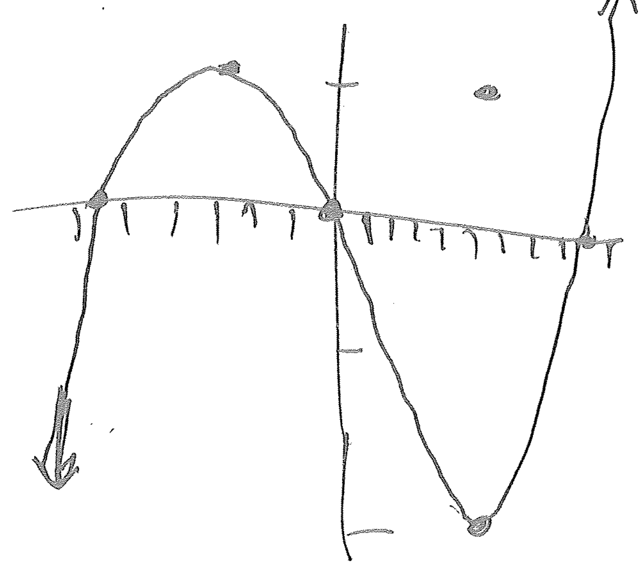
$f''(-3) = -24$ $\quad$ CDN $(-\infty, 1)$

$f''(5) = 24$ $\quad$ cup $(1, \infty)$

$f(-3) = 81$ lmax

$f(5) = -175$ lmin

$f(1) = -47$ ip



7. (25 pts) Use the fact that

$$\frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x \cot x$$

to prove that

$$\frac{\mathrm{d}}{\mathrm{d}x}\csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$$

$y = \csc^{-1} x$

$\csc y = x$

$-\csc y \cot y \cdot y' = 1$

$y' = \dfrac{-1}{\csc y \cot y}$

$= \dfrac{-1}{\csc \csc^{-1} x \cot \csc^{-1} x}$

$= \dfrac{-1}{x \pm \sqrt{\csc^2 \csc^{-1} x - 1}} = \dfrac{-1}{|x|\sqrt{x^2-1}}$

$\cot^2 \alpha + 1 = \csc^2 \alpha$

$\cot \alpha = \pm\sqrt{\csc^2 \alpha - 1}$

8. (25 pts) Find

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\csc^{-1} x\right)^{(\csc x)}$$

$$y = \csc^{-1} x^{\csc x}$$

$$\ln y = \csc x \ln \csc^{-1} x$$

$$\frac{y'}{y} = -\csc x \cot x \ln \csc^{-1} x + \csc x \frac{\frac{-1}{|x|\sqrt{x^2-1}}}{\csc^{-1} x}$$

$$y' = \left(-\csc x \cot x \ln \csc^{-1} x - \frac{\csc x}{|x|\sqrt{x^2-1}\csc^{-1} x}\right) \csc^{-1} x^{\csc x}$$

9. (25 pts) Approximate $\sqrt[4]{15}$ using only elementary arithmetic operations with each of the following methods:

(a) Local linear approximation

$f(x) = \sqrt[4]{x}$ $a = 16$

$f'(x) = \frac{1}{4}x^{-3/2}$

$$L(x) = f(a) + f'(a)(x-a)$$
$$= 2 + \frac{1}{4}(16)^{-3/4}(-1)$$
$$= 2\overline{0}\,{}^1/_{32} = \cancel{2.03125}$$

1.96875

(b) Five iterations of bisection starting on the interval [1, 2]

$f(x) = x^4 - 15$

a	$\frac{a+b}{2}$	b	$f(a)$	$f(\frac{a+b}{2})$	$f(b)$
1	1.5	2	−14	−9.94	1
1.5	1.75	2	−9.94	−5.6	1
1.75	1.875	2	−5.6	−2.64	1
1.875	1.9375	2	−2.64	−0.91	1
1.9375	1.96875	2	−0.91	0.02	1

$1.9375 < \sqrt[4]{15} < 1.96875$ $\sqrt[4]{15} \approx 1.953125$

(c) Five iterations of Newton's method starting with $x_0 = 2$

$x_0 = 2$

$$x_1 = 2 - \frac{2^4 - 15}{4(2^3)} = 1.96875$$

$$x_2 = 1.96875 - \frac{1.96875^4 - 15}{4(1.96875)^3} = 1.967940112$$

$x_3 = 1.967989671$

$x_5 = x_4 = x_3$

10. (25 pts) Find how quickly the surface area of a cube is increasing when its volume is increasing at a rate of $1\frac{\text{in}^3}{\text{s}}$. and the sides are 1 in long. (sorry

$V = x^3$

$A = 6x^2$

$A' = 12x\,x'$

$V'/3x^2 = x'$

$1 = V' = 3x^2 x'$

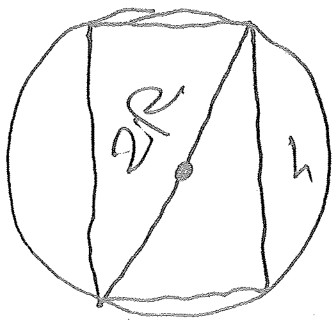
$1/3x^2 = x'$

$A' = 12x/(3x^2)$

$= \frac{4}{x}$

$A' = \frac{4}{1} = 4 \text{ in}^2/\text{s}.$

11. (50 pts) Find the volume of the largest cylinder that can be contained inside a sphere of radius R.



$V = \pi r^2 h$

dom $V = [0, 2R]$

$(2r)^2 + h^2 = (2R)^2$

$4r^2 + h^2 = 4R^2$

$r^2 = R^2 - \frac{h^2}{4}$

$V = \pi\left(R^2 - \frac{h^2}{4}\right)h$

$V = \pi R^2 h - \pi h^3/4$

$0 = V' = \pi\left(R^2 - 3h^2/4\right)$

$h = \sqrt{4R^2/3} = 2R/\sqrt{3}$

$V(0) = 0$

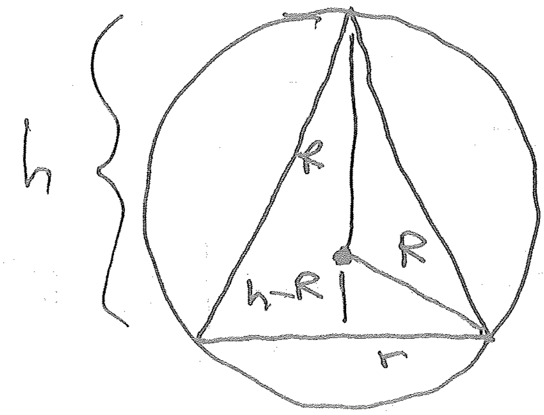
$V(2R/\sqrt{3}) =$

$\pi\left(R^2 - \frac{4R^2/3}{4}\right) 2R/\sqrt{3}$

$= \pi\left(2R^2/3\right) 2R/\sqrt{3}$

$= \pi 4R^3/3\sqrt{3}$

12. (50 pts) Find the volume of the largest cone that can be contained inside a sphere of radius R.



$V = \frac{1}{3}\pi r^2 h$

$(h-R)^2 + r^2 = R^2$

$(R-h)^2 + r^2 = R^2$

$\cancel{R^2} - 2hR + h^2 + r^2 = \cancel{R^2}$

$r^2 = 2hR - h^2$

$V = \frac{1}{3}\pi(2hR - h^2)h$

$V = \frac{1}{3}\pi(2Rh^2 - h^3)$

$0 = V' = \frac{1}{3}\pi(4Rh - 3h^2) = \frac{1}{3}\pi h(4R - 3h)$

$h = 0,\ 4R/3$

dom $V = [0, 2R]$

$V(0) = 0$

$V(4R/3) = \frac{1}{3}\pi\left(2\frac{4R}{3}R - \left(\frac{4R}{3}\right)^2\right)\cdot\frac{4R}{3}$

$= \frac{1}{3}\pi\left(\frac{8R^2}{3} - \frac{16R^2}{9}\right)\frac{4R}{3}$

$= \frac{1}{3}\pi\left(\frac{8R^2}{9}\right)\frac{4R}{3}$

$= \boxed{\frac{32\pi R^3}{81}}$ ← max

$V(2R) = 0$