Final Exam

Name: ______________________

The optimization problems are worth 50 points; each of the other problems is worth 25 points. Work at least one of the optimization problems and all of the remaining problems; you may work all twelve problems for extra credit. Numerical estimates are unacceptable unless specifically requested; for full credit you must show all your work and use the indicated methods.

1. (25 pts) Use the definition of the limit to show that

$$\lim_{x \to -3} (-5x + 3) = 18$$

2. (25 pts) Find

$$\lim_{x \to 0^+} \left(\ln x + \frac{1}{x} \right)$$

3. (25 pts) Find

$$\lim_{t \to 1} \frac{e^{3t} - e^3}{e^{2t} + e^t + 1}$$

4. (25 pts) Use the definition of continuity to determine where $f(x)$ is continuous, where

$$f(x) = \begin{cases} x^2 + 5 & x > 4 \\ -3x + 33 & x \leq 4 \end{cases}$$

5. (25 pts) Use the definition of the derivative to find

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sqrt{x}}$$

6. (25 pts) Sketch the graph of $f(x)$ by hand, where

$$f(x) = x^3 - 3x^2 - 45x$$

Show all intercepts, asymptotes, extrema, and inflection points.

7. (25 pts) Use the fact that

$$\frac{\mathrm{d}}{\mathrm{d}x} \csc x = -\csc x \cot x$$

to prove that

$$\frac{\mathrm{d}}{\mathrm{d}x} \csc^{-1} x = -\frac{1}{x\sqrt{x^2 - 1}}$$

8. (25 pts) Find

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\csc^{-1} x\right)^{(\csc x)}$$

9. (25 pts) Approximate $\sqrt[4]{15}$ using only elementary arithmetic operations with each of the following methods:

   (a) Local linear approximation

   (b) Five iterations of bisection starting on the interval $[1, 2]$

   (c) Five iterations of Newton's method starting with $x_0 = 2$

10. (25 pts) Find how quickly the surface area of a cube is increasing when its volume is increasing at a rate of $1\frac{\text{in}^3}{\text{s}}$.

11. (50 pts) Find the volume of the largest cylinder that can be contained inside a sphere of radius R.

12. (50 pts) Find the volume of the largest cone that can be contained inside a sphere of radius R.