Exam 4

Name: KEY

The optimization problem is worth 55 points; each of the other problems is worth 15 points. Work the last problem and at least three of the first four problems; you may work all five problems for extra credit. Numerical estimates are unacceptable; unless specifically requested for full credit you must show all your work and use the indicated methods.

1. (15 pts) Find all maxima, minima, and inflection points of:

$$f(x) = e^{-4x^2+8x}$$

$$f'(x) = (-8x+8)e^{-4x^2+8x}$$

$x = 1$ cpt

$$f''(x) = -8e^{-4x^2+8x} + (-8x+8)^2 e^{-4x^2+8x}$$

$$= \cancel{-8xe^{-4x^2+8x}} \quad (64x^2 - 128x + 56)e^{-4x^2+8x}$$

$$= 8(8x^2 - 16x + 7)e^{-4x^2+8x}$$

$$\frac{16 \pm \sqrt{(-16)^2 - 4(8)(7)}}{2(8)} = \frac{16 \pm \sqrt{32}}{16}$$

$$= 1 \pm \sqrt{2}/4$$

$(-8x+8)^2 = 8$

$-8x+8 = \pm 2\sqrt{2}$

$-8x = -8 \pm 2\sqrt{2}$

$x = 1 \pm \sqrt{2}/4$

possible ips @ $x =$

$f''(1) = -8e^{-4} < 0$

So $x=1$ lmax

$f''(0) = 56e^{0}$

$f''(1) = -8e^{-4}$

$f''(2) = 56e^{0}$

$f''(-1) = 8e^{-4-8} > 0$

$f(1-\sqrt{2}/4) =$

So $x=0$ is ip

$f(0) = 1$ ip

$f(1) = e^4$ lmax, gmax (only one)

$x = 1 \pm \sqrt{2}/4$

ips

2. (15 pts) Find

$$\lim_{x \to 0} \frac{\tan^{-1} x}{e^x - 1}$$

$0/0$

$$= \lim_{x \to 0} \frac{\frac{1}{x^2+1}}{e^x} = \frac{1}{1} = 1$$

$y = \tan^{-1}$

$x = \tan y$

$1 = y' \sec^2 y$

$$y' = \frac{1}{\sec^2 y}$$

$$= \frac{1}{\sec^2(\tan^{-1} x)}$$

$$= \frac{1}{\tan^2 \tan^{-1} x + 1}$$

$$= \frac{1}{x^2+1}$$

3. (15 pts) Find

$$\lim_{x\to\infty} (\ln x)^{(1/\ln x)}$$

$y = (\ln x)^{1/\ln x}$

$\ln y = \frac{1}{\ln x} \ln \ln x$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln \ln x}{\ln x} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{\frac{1}{x\ln x}}{\frac{1}{x}}$

$= \lim_{x\to\infty} \frac{1}{\ln x} = 0$

So $\lim_{x\to\infty} y = 1$

4. (15 pts) Use Newton's method to approximate $\sqrt{2}$ using only elementary arithmetic operations. Write out the complete statement of Newton's method for the first two approximations after your initial guess, and write out the value of all successive approximations until the sequence converges to within the maximum accuracy of your calculator.

$x_0 = 2 \qquad f(x) = x^2 - 2 \qquad f'(x) = 2x$

$$0 = L(x) = f'(x_n)(x_{n+1} - x_n) + f(x_n)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n}$$

$$= \frac{2x_n^2 - x_n^2 + 2}{2x_n}$$

$$= \frac{x_n^2 + 2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}$$

$x_1 = \frac{2}{2} + \frac{1}{2} = 1.5$

$x_2 = \frac{1.5}{2} + \frac{1}{1.5} = 1.4167$

$x_3 = 1.414215686$

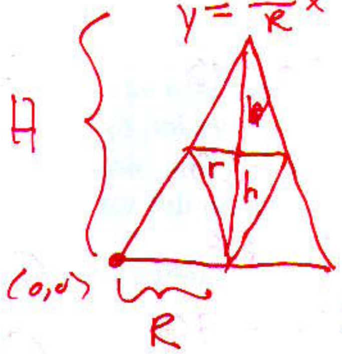
$x_4 = 1.414213562$

$x_5 = 1.414213562$

5. (55 pts) Suppose a smaller cone is inverted and placed inside a larger cone so that the bases of the two cones are parallel. Find the largest possible volume of the smaller cone if the larger cone has radius $R$ and height $H$.

Use that $y = \frac{H}{R}x$

$V = \frac{1}{3}\pi r^2 h$




$\cancel{\frac{H}{R}(R-r) = H-h}$

$$\frac{r}{H-h} = \frac{R}{H}$$

$$H-h = \frac{rH}{R} \qquad h = H - \frac{rH}{R}$$

$$V = \frac{1}{3}\pi r^2 \left(H - \frac{rH}{R}\right) = \cancel{\frac{1}{3}\pi r^3 H / R}$$

$$V = \frac{1}{3}\pi r^2 H \cdot \left(1 - \frac{r}{R}\right)$$

$$\cancel{V' = \frac{\pi r^2 H}{R}} = \frac{1}{3}\pi r^2 H - \frac{\pi r^3 H}{3R}$$

$$V' = \frac{2}{3}\pi r H - \frac{\pi r^2 H}{R} = \pi r H\left(\frac{2}{3}\right.$$

$$0 = \frac{2}{3}\pi r H - \frac{\pi r^2 H}{R} \qquad \mathrm{dom}\, V = [0, R]$$

$$= \pi r H\left(\frac{2}{3} - \frac{r}{R}\right)$$

$h =$

$\frac{H}{3} =$

$$r = 0, \quad \frac{2}{3}R = r$$

$$h = H - \frac{\left(\frac{2}{3}R\right)H}{R}$$

$$V'' = \frac{2}{3}\pi H - \frac{2\pi r H}{R} \qquad = H - \frac{2}{3}H = \frac{H}{3} = h$$

$$V(0) = 0$$

$$V\left(\frac{2}{3}R\right) = \frac{1}{3}\pi\left(\frac{2}{3}R\right)^2 H\left(1 - \frac{2/3\, R}{R}\right)$$

$$V(R) = 0 \qquad = \frac{4\pi R^2 H}{81} = V$$