Name: KEY

Each problem is worth 15 points. Numerical estimates are unacceptable; for full credit you must show all your work and use the indicated methods.

1. Suppose the position of a particle after t seconds is given by $f(t) = t^3 - 3t^2 + 2t + 1$. Determine when the particle is speeding up and when it is slowing down.

$f'(t) = 3t^2 - 6t + 2$

$f''(t) = 6t - 6$

$0 = 3t^2 - 6t + 2$

$t = \dfrac{6 \pm \sqrt{(-6)^2 - 4(3)(2)}}{2(3)} = \dfrac{6 \pm \sqrt{36-24}}{6} = 1 \pm \dfrac{\sqrt{12}}{6}$

$= 1 \pm \sqrt{3}/3$

$\cong 0.423, 1.577$

$0 = 6t - 6$

$t = 1$

		f'	f''	
$(0, 1-\sqrt{3}/3)$	$\cong (0, 0.423)$	$+$	$-$	slowing
$(1-\sqrt{3}/3, 1)$	$\cong (0.423, 1)$	$-$	$-$	speeding
$(1, 1+\sqrt{3}/3)$	$(1, 1.577)$	$-$	$+$	slowing
$(1+\sqrt{3}/3, \infty)$	$(1.577, \infty)$	$+$	$+$	speeding

2. Find the equation for the highest horizontal tangent line to

$$f(x) = x^2 e^x$$

$f'(x) = 2xe^x + x^2 e^x$

$= e^x(x^2 + 2x)$

$0 = e^x(x^2 + 2x)$

$x^2 + 2x = 0$

$x(x+2) = 0$

$x = 0, -2$

$f(0) = 0$

$f(-2) = 4e^{-2} > 0$

So $\boxed{y = 4e^{-2}}$

3. Use the fact that $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = -\sin x$ to prove that $\frac{d}{dx}\tan x = \sec^2 x$.

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

4. Use the definition of the derivative to find

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{(4x+3)^2}$$

$$\lim_{h\to 0}\frac{\frac{1}{(4(x+h)+3)^2}-\frac{1}{(4x+3)^2}}{h}$$

$$=\lim_{h\to 0}\frac{(4x+3)^2-(4(x+h)+3)^2}{h(4x+3)^2(4(x+h)+3)^2}$$

$$=\lim_{h\to 0}\frac{16x^2+24x+9-16(x+h)^2-24(x+h)-9}{h(4x+3)^2(4(x+h)+3)^2}$$

$$=\lim_{h\to 0}\frac{\cancel{16x^2}+\cancel{24x}+\cancel{9}-\cancel{16x^2}-32xh-16h^2-\cancel{24x}-24h-\cancel{9}}{h(4x+3)^2(4(x+h)+3)^2}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(-32x-16h-24)}{\cancel{h}(4x+3)^2(4(x+h)+3)^2}=\lim_{h\to 0}\frac{-32x-16h-24}{(4x+3)^2(4(x+h)+3)^2}$$

$$=\frac{-32x-24}{(4x+3)^4}=\frac{-8(4x+3)}{(4x+3)^4}=\frac{-8}{(4x+3)^3}$$

5. Suppose the temperature (in °C) of a point on an iron rod protruding from a kiln is given as a function of the distance (in cm) to the end by

$$T(d) = \begin{cases} 1500 & \text{if } 0 \leq \cancel{x}\,d < 16 \\ \frac{3000\sqrt{x}}{\sqrt[4]{x^3}} & \text{if } \cancel{x}\,d \geq 16 \end{cases} = 3000/x^{1/4}$$

Find $T'(81)$. Give units and interpret your answer.

For $x \geq 16$

$$T'(d) = -\frac{3000}{4} x^{-5/4}$$

$$T'(81) = -3.086 \ °C/cm$$

81 cm from the end of the rod, temperature is dropping at a rate of 3.086 °C/cm

6. The functions f, f', and f'' are pictured below. Label each graph with the appropriate function. Justify your answer by identifying how sign, slope and concavity correspond between the graphs of the function and its derivatives.