Final Exam

Name: KEY

1. Use the definition of the limit to show that

$$\lim_{x \to 0+} \sqrt{x} = 0$$

Let $\varepsilon > 0$, set $\delta = \varepsilon^2$.

Then

$0 < |x - 0| < \delta = \varepsilon^2$, $x > 0 \Rightarrow$

$0 < x < \delta = \varepsilon^2 \Rightarrow$

$\sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = |\varepsilon| = \varepsilon \Rightarrow$

$\sqrt{x} - 0 < \varepsilon \Rightarrow$

$|\sqrt{x} - 0| < \varepsilon$ $(\because \sqrt{x} - 0 \geq 0)$

2. Find

$$\lim_{x\to\infty}\left(\frac{e^{3x}-1}{e^{(x^2)}+5}\right)$$

∞/∞

$$= \lim_{x\to\infty} \frac{3e^{3x}}{2xe^{x^2}} = \lim_{x\to\infty} \frac{3}{2xe^{x^2-3x}}$$

But $\lim_{x\to\infty} x^2-3x = \lim_{x\to\infty} x(x-3) = \infty\cdot\infty = \infty$

So $\lim_{x\to\infty} \frac{3}{2xe^{x^2-3x}} = \frac{3}{2\infty e^{\infty}} = \frac{3}{\infty} = 0$

3. Show that
$$\lim_{x\to 0}\left(x^2\sin\left(\frac{\pi}{x}\right)\right)=0$$

$$-1 \le \sin\frac{\pi}{x} \le 1$$

$$-x^2 \le x^2\sin\frac{\pi}{x} \le x^2$$

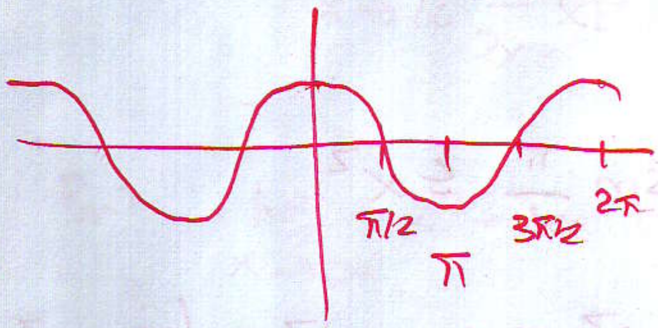
$$\lim_{x\to 0} -x^2 = -0^2 = 0 = 0^2 = \lim_{x\to 0} x^2$$

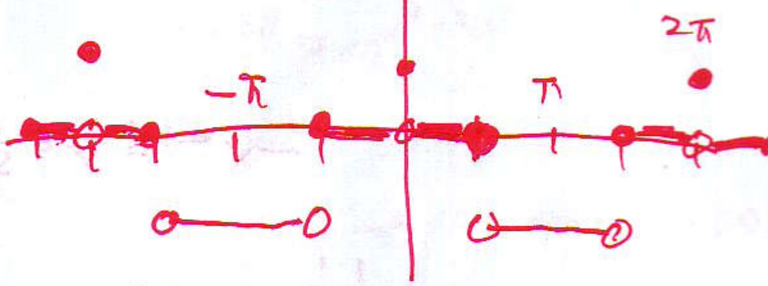
So $\lim_{x\to 0} x^2 \sin\frac{\pi}{x} = 0$ by the Squeeze Thm.

4. List all discontinuities of

$$f(x) = [\![\cos x]\!]$$

(Hint: graph $\cos x$ and $[\![\cos x]\!]$ by hand, then list the discontinuities on your graph).





Discontinuous at odd multiples of $\pi/2$ & even multiples of π:

$\ldots, -3\pi/2, -\pi/2, \pi/2, 3\pi/2, \ldots$

$\ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$

So discont's @

$x = (2k+1)\pi/2, \ k \in \mathbb{Z}$ and

$2k\pi, \ k \in \mathbb{Z}$

5. Use the definition of the derivative to find

$$\frac{d}{dx}\left(x^2 - 3x + 5\right)$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - 3(x+h) + 5 - (x^2 - 3x + 5)}{h}$$

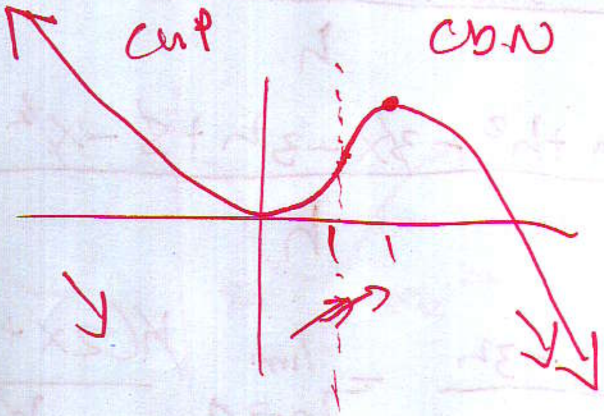
$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{3x} - 3h + \cancel{5} - \cancel{x^2} + \cancel{3x} - \cancel{5}}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - 3h}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x + h - 3)}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h - 3 = 2x - 3$$

6. Draw a possible graph of $f(x)$, assuming the following:

- $f'(x)$ is positive on $[0, 2]$ and negative everywhere else
- $f'(x)$ is increasing when $x < 1$ and decreasing when $x > 1$



7. State and prove the differentiation rule for constant functions.

$\frac{d}{dx} c = 0$ i.e. if $f(x) = c$, $f'(x) = 0$

Pf: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$

$\lim_{h \to 0} \frac{c-c}{h} = \lim_{h \to 0} \frac{0}{h}$

$= \lim_{h \to 0} 0 = 0$

8. Suppose the distance (in feet) from a beach house to the water line t hours after sunrise is given by

$$D(t) = 40 + 20\cos\left(\frac{2\pi}{12}t\right)$$

Find the rate at which the water line is moving at noon, assuming sunrise is at 6:30 a.m.

$$D'(t) = -20\sin\left(\frac{2\pi}{12}t\right)\left(\frac{2\pi}{12}\right)$$

$$D'(5.5) = -20\sin\left(\frac{2\pi}{12}5.5\right)\left(\frac{2\pi}{12}\right)$$

$$= -2.71$$

The water line is receding

by 2.71 ft/hr at noon.

9. Find y' if

$$e^y = e^x$$

$$(e^y)y' = e^x$$

$$y' = \frac{e^x}{e^y} = e^{x-y}$$

10. Find

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec x)^{(\csc x)}$$

$$y = (\sec x)^{(\csc x)}$$

$$\ln y = \csc x \ln(\sec x)$$

$$\frac{y'}{y} = -\csc x \cot x \ln(\sec x) + \frac{\csc x \cancel{\sec x} \tan x}{\cancel{\sec x}}$$

$$= -\csc x \cot x \ln(\sec x) + \csc x \tan x$$

$$= \csc x(\tan x - \cot x \ln(\sec x))$$

So $y' = \csc x(\tan x - \cot x \ln(\sec x))(\sec x)^{(\csc x)}$

11. Approximate $e^{1/10}$ using local linear approximation. Perform all arithmetic by hand, and use a graph to illustrate whether your answer is an overestimate or an underestimate.

$a = 0$

$x = 1/10$

$f(x) = e^x$

$f'(x) = e^x$

$f(a) = f'(a) = e^0 = 1$

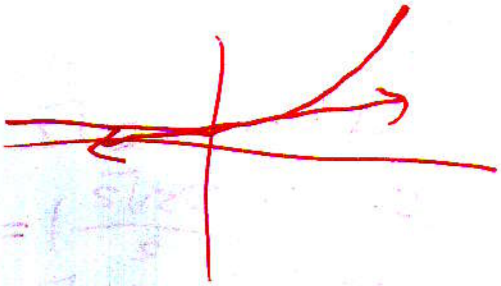
$L(x) = f'(a)(x-a) + f(a)$

$= 1(x - 0) + 1$

$= x - 0 + 1$

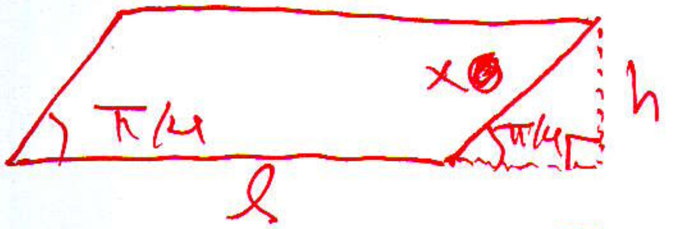
$= x + 1$

$L(1/10) = 1.1$



underest.

12. Find the dimensions of the parallelogram of perimeter 100 and base angle $\frac{\pi}{4}$ with maximum internal area. Use the fact that the area of a parallelogram is given by its width times its height.



$P = 100$ $\qquad \frac{\sqrt{2}}{2} = \sin\frac{\pi}{4} = \frac{h}{x}$

$P = 2l + 2x$ $\qquad x = \frac{h}{\sin\frac{\pi}{4}} = \frac{h}{\sqrt{2}/2}$

$100 = 2l + 2x$ $\qquad = h\sqrt{2}$

$= 2l + 2\sqrt{2}h$

So $l = \frac{100 - 2\sqrt{2}h}{2} = 50 - \sqrt{2}h$

$h > 0 \quad l > 0 \Rightarrow 50 - \sqrt{2}h > 0$
$\Rightarrow \sqrt{2}h < 50 \Rightarrow$
$h < 50/\sqrt{2} = 25\sqrt{2}$

dom $A = (0, 25\sqrt{2})$

$A = lh$

$= (50 - \sqrt{2}h)h$

$= 50h - \sqrt{2}h^2$ $\qquad A'' = -2\sqrt{2}$

$0 = A' = 50 - 2\sqrt{2}h$ $\qquad$ so $A''\left(\frac{25\sqrt{2}}{2}\right) = -2\sqrt{2} < 0$

$50 = 2\sqrt{2}h$ $\qquad \Rightarrow h = \frac{25\sqrt{2}}{2}$ lmax $\Rightarrow$

$h = \frac{50}{2\sqrt{2}} = \frac{25\sqrt{2}}{2}$. $\qquad \boxed{h = \frac{25\sqrt{2}}{2}}$ gmax

(only CN)

$\boxed{l = 50 - \sqrt{2}h = 50 - 25 = 25}$