/

Exam 2

Name: KEY

Each problem is worth 30 points. Complete at least five problems; you may complete the sixth for extra credit. Show all your work.

1. State the constant multiple rule, and prove the rule using the definition of the derivative.

$$\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}f(x)$$

Pf: $\frac{d}{dx}\left[cf(x)\right] \overset{\text{def}}{=} \lim_{h\to 0} \frac{cf(x+h)-cf(x)}{h}$

$= \lim_{h\to 0} c\frac{f(x+h)-f(x)}{h} \overset{\text{limit laws}}{=} c\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$\overset{\text{def}}{=} c\frac{d}{dx}f(x)$

2. Find $\frac{d}{dx}\frac{\sqrt[3]{x}}{\sqrt[3]{x^7}}$ using the definition of the derivative. Check your work using differentiation rules.

$$\frac{\sqrt[3]{x}}{\sqrt[3]{x^7}} = x^{1/3-7/3} = x^{-6/3} = x^{-2}$$

$$\frac{d}{dx}x^{-2} = \lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} = \lim_{h\to 0}\frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h}$$

$$= \lim_{h\to 0}\frac{x^2-x^2-2xh-h^2}{hx^2(x+h)^2} = \lim_{h\to 0}\frac{-2xh-h^2}{hx^2(x+h)^2} = \lim_{h\to 0}\frac{-2x-h}{x^2(x+h)^2} =$$

$$\frac{-2x}{x^4} = \frac{-2}{x^3}$$

$$\frac{d}{dx}x^{-2} = -2x^{-3} = -\frac{2}{x^3}$$

3. Approximate the slope of the graph of $f(x) = \sin(\pi \sin(x))$ at $x = \pi$, and use your approximation to find a formula for the tangent line to the graph of $f(x)$ at $x = \pi$.

x	$\frac{\sin(\pi \sin(x)) - \sin(\pi \sin(\pi))}{x - \pi}$
$\pi - 1$	-0.4777
$\pi - 0.1$	-3.0852
$\pi - 0.01$	-3.14102
$\pi + 0.01$	-3.14102
$\pi + 0.1$	-3.0852
$\pi + 1$	-0.4777

$f'(\pi) \simeq -\pi$

$f(\pi) =$
$\sin(\pi \sin \pi)$
$= \sin(0)$
$= 0$

$y - 0 = -\pi(x - \pi)$
$= -\pi x + \pi^2$

4. Find

$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}e^x$$

$f = \sqrt{x} = x^{1/2}$ $\qquad f' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$g = e^x$ $\qquad g' = e^x$

$$f'g + fg' = \frac{e^x}{2\sqrt{x}} + \sqrt{x}e^x$$

$$= e^x\left(\frac{1}{2\sqrt{x}} + \sqrt{x}\right)$$

5. Suppose the number of UT students infected with a mutant strain of the H5N1 virus t weeks after delivery of an contaminated shipment of poultry is given by

$$I(t) = \frac{25,000t^2}{50 + t^2} = \frac{f}{g}$$

Determine when the infection is speeding up and slowing down.

$f' = 50,000t$

$g' = 2t$

$$I' = \frac{f'g - fg'}{g^2}$$

$$= \frac{50,000t(50+t^2) - 25,000t^2(2t)}{(50+t^2)^2}$$

$$= \frac{2,500,000t + 50,000t^3 - 50,000t^3}{t^4 + 100t^2 + 2500}$$

$$= \frac{2,500,000t}{t^4 + 100t^2 + 2500} = \frac{h}{k}$$

$I' > 0$ for $t > 0$, I always increases

Speeding up for $0 \le t \le 4.08$, slowing down afterward

Also $k = (t^2 + 50)^2$

$h' = 2,500,000$

$k' = 4t^3 + 200t$

$$I'' = \frac{h'k - hk'}{k^2}$$

$$= \frac{2,500,000(t^4 + 100t^2 + 2500) - 2,500,000t(4t^3 + 200t)}{(t^2 + 50)^4}$$

$$= (2,500,000t^4 + 250,000,000t^2 + 6,250,000,000 - 10,000,000t^4 - 500,000,000t^2) / (t^2+50)^4$$

$$= -\frac{7,500,000t^4 - 250,000,000t^2 \cancel{- 500,000,000t} + 6,250,000,000}{(t^2+50)^4}$$

$$= -\frac{2,500,000(\cancel{3t^4 + 100t^3 - 200t + 1500})(3t^4 + 100t^2 - 2500)}{(t^2+50)^4}$$

$(t^2+50)^4$ ← Always positive

$3t^4 + 100t^2 - 2500 = 0$

$u = t^2 \quad (u > 0)$

$3u^2 + 100u - 2500 = 0$

$$u = \frac{-100 \pm \sqrt{100^2 + 30,000}}{6}$$

$= \cancel{-50}, 16.67$

$t = \pm\sqrt{16.67}$

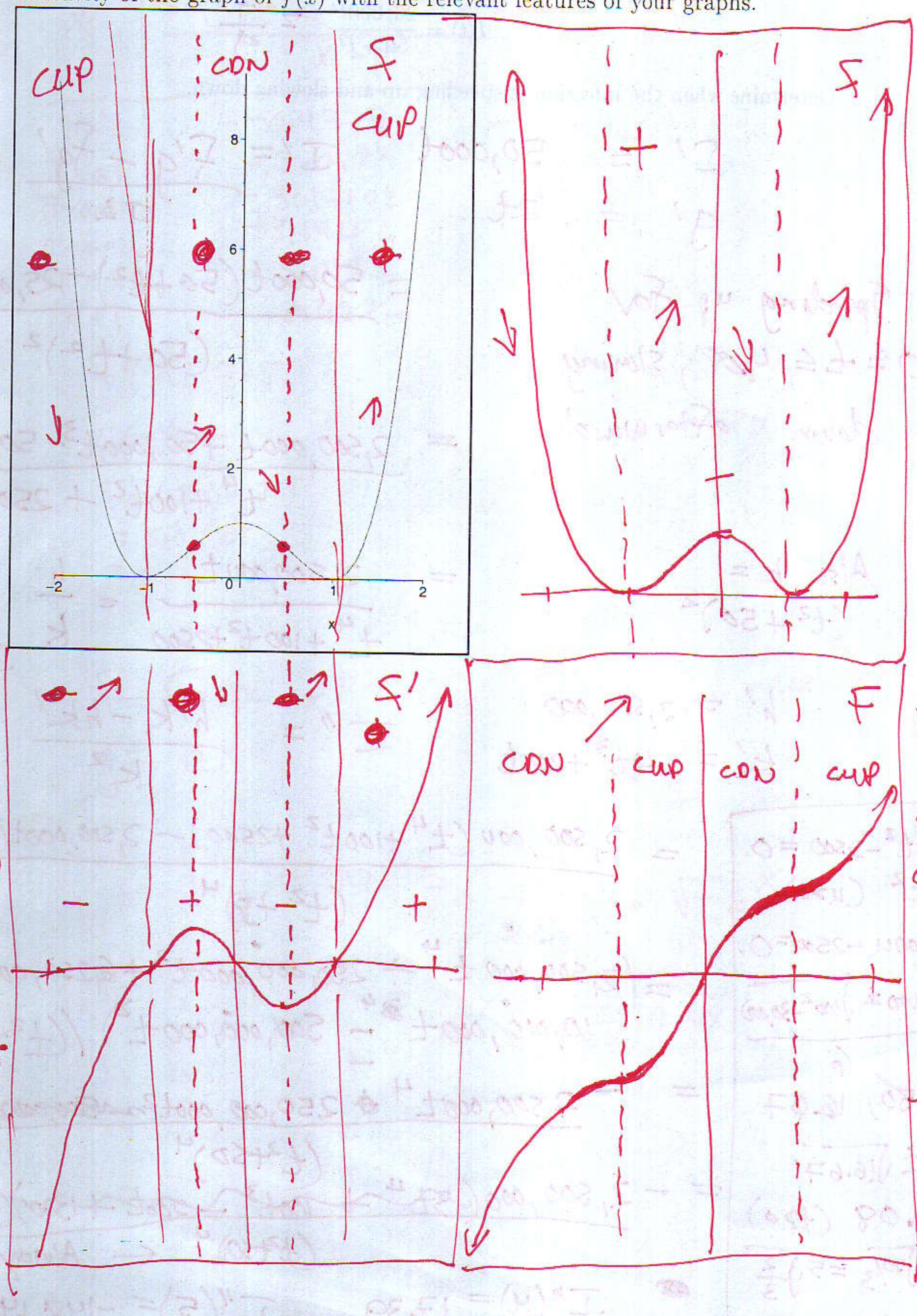
$\approx 4.08 \quad (t > 0)$

$c = \sqrt{50/3} = 5\sqrt{\frac{2}{3}}$

$I''(4) = 17.39 \qquad I''(5) = -148,148$

When the man gets round,
I won't be found.
I'm an elsewere.

6. For the function $f(x)$ pictured below, sketch $f'(x)$ and a possible antiderivative $F(x)$. Justify each of your graphs by writing a sentence relating the sign, slope, and concavity of the graph of $f(x)$ with the relevant features of your graphs.



$f(x)$ is dec. from $(-2,-1)$, $(0,1)$ and inc. elsewhere, so f' is − from $(-2,-1)$, $(0,1)$ and + elsewhere

f' dec where f is down; f' inc where f is cup

$f>0$ $\forall x$, so F inc. $\forall x$

F is CDN where f dec; F is cu[p] where F inc

F levels off where $f=0$