In section 3.8 we saw how to use tangent lines to approximate the values of irrational numbers such as $\sqrt{7}$ using a tangent line to the function $f(x)$ at a point $x = a$ where the function may be easily evaluated (i.e. without a calculator). In all of the examples done in class, the point $x = a$ was a perfect square integer. However, we can attain a better estimate by letting $x = a$ be a rational number and a perfect square, since some of these values will be closer to 7 than any integer perfect square. To find such a value, we make a table with n^2 and $7n^2$ for all $1 \leq n \leq 20$:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
n^2	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225	256	289	324	361	400
$7n^2$	7	28	63	112	175	252	343	448	567	700	847	1008	1183	1372	1575	1792	2023	2268	2527	2800

When one of the numbers $7n^2$ in the last row is almost equal to n^2 in the second row, the ratio of the two numbers in the second row will be a perfect square extremely close to 7. From the table above, we see that $\left(\frac{13}{5}\right)^2 = \frac{169}{25} < \frac{175}{25} = 7 = \frac{63}{9} < \frac{64}{9} = \left(\frac{8}{3}\right)^2$. So as an initial estimate, $\frac{13}{5} < \sqrt{7} < \frac{8}{3}$, or $2.6 < \sqrt{7} < 2.\overline{6}$. To make this estimate more precise, we will compute the tangent line approximation at $a = \frac{169}{25}$ and $b = \frac{64}{9}$ and the secant line approximation between a and b.

<u>Tangent line approximation at $a = \frac{169}{25}$:</u>

$$\begin{aligned}
f(x) &\approx \frac{1}{2\sqrt{\frac{169}{25}}}\left(7 - \frac{169}{25}\right) + \sqrt{\frac{169}{25}} \\
&= \frac{1}{2\left(\frac{13}{5}\right)} \frac{175 - 169}{25} + \frac{13}{5} \\
&= \frac{5}{26}\left(\frac{6}{25}\right) + \frac{13}{5} \\
&= \frac{6}{130} + \frac{13}{5} \\
&= \frac{6 + 338}{130} \\
&= \frac{344}{130} \\
&= 2 + \frac{84}{130} \\
&= 2 + \frac{42}{65} \\
&= 2.64\overline{615384}
\end{aligned}$$

Tangent line approximation at $b = \frac{64}{9}$:

$$\begin{aligned}
f(x) &\approx f'(a)(x-a) + f(a) \\
&= \frac{1}{2\sqrt{\frac{64}{9}}}\left(7 - \frac{64}{9}\right) + \sqrt{\frac{64}{9}} \\
&= \frac{1}{2\left(\frac{8}{3}\right)} \frac{63-64}{9} + \frac{8}{3} \\
&= \frac{3}{16}\left(\frac{-1}{9}\right) + \frac{8}{3} \\
&= \frac{-1}{48} + \frac{8}{3} \\
&= \frac{-1+128}{48} \\
&= \frac{127}{48} \\
&= 2 + \frac{31}{48} \\
&= 2.6458\overline{3}
\end{aligned}$$

Since the graph of $f(x) = \sqrt{x}$ is concave down, the tangent line approximations are both overestimates, so we will pick the smaller one as the better estimate: $\sqrt{7} < 2.6458\overline{3}$. However, this isn't hugely useful since we still don't have any idea how much less than our estimate the exact value is. To get a better idea of how close we are without actually computing the exact value of $\sqrt{7}$, we will use a secant line approximation to get an underestimate. The setup here is identical to the tangent line approximation, except that we use the slope of the secant line $\frac{f(b)-f(a)}{b-a}$ in place of $f'(a)$ or $f'(b)$. This gives

Secant line approximation from $a = \frac{169}{25}$ to $b = \frac{64}{9}$:

$$\begin{aligned}
f(x) &\approx \frac{f(b)-f(a)}{b-a} + f(a) \\
&= \frac{\sqrt{\frac{64}{9}} - \sqrt{\frac{169}{25}}}{\frac{64}{9} - \frac{169}{25}}\left(7 - \frac{64}{9}\right) + \sqrt{\frac{64}{9}} \\
&= \frac{\frac{8}{3} - \frac{13}{5}}{\frac{64}{9} - \frac{169}{25}}\left(\frac{-1}{9}\right) + \frac{8}{3} \\
&= \frac{\frac{1}{15}}{\frac{79}{225}}\left(\frac{-1}{9}\right) + \frac{8}{3} \\
&= \frac{-5}{237} + \frac{8}{3} \\
&= \frac{209}{79} \\
&= 2 + \frac{51}{79} \\
&= 2.\overline{6455696202531}
\end{aligned}$$

Since the graph of $f(x) = \sqrt{x}$ is concave down, the secant line approximation is an underestimate, so we now know that $2.\overline{6455696202531} < \sqrt{7} < 2.6458\overline{3}$. So we can reasonably state that when rounding to the third digit after the decimal, $\sqrt{7} = 2.646$ since both the overestimate and the underestimate round to this same value. Note that given enough time, all of these computations could be performed entirely by hand without using a calculator. However, I will only expect to see the computation of square roots by hand in the assignment.

To find a more nearly exact value, we will apply Newton's method to this initial guess of 2.646:

x_0	2.646
x_1	2.645751323
x_2	2.645751311
x_3	x_2

This approximates $\sqrt{7}$ to 9 decimals, and shows that the initial overestimates and underestimate were correct.

<u>Exercises</u>

Complete <u>either</u> of the exercises below for 15 points of extra credit (if you do both, you will still only get 15 points). In either exercise, the values of a and b that you pick must be within 0.5 units of 11 (or 13 for #2), and you must choose a and b so that $a < 11 < b$ (or $a < 13 < b$ for #2). Please note that while I want to see everything worked out as above until the second or third step (whichever has no square roots left), you do not need to show simplifying the fractions by hand.

1. Use the above methods to find two overestimates and an underestimate for $\sqrt{11}$, and use Newton's method to check your work.

2. Use the above methods to find two overestimates and an underestimate for $\sqrt{13}$, and use Newton's method to check your work.