Name: ______________________

Each problem is worth 25 points. Show all your work for full credit; numerical or graphical estimates are unacceptable unless specifically requested. Work at least 12 problems; you may work all 13 for extra credit.

1. Use the $\epsilon - \delta$ definition of the limit to prove that for $f(x) = 11x + 10$,

$$\lim_{x \to -2} f(x) = -12$$

Draw a graph showing what a, δ, L, and ϵ mean for this problem.

2. Find

$$\lim_{x \to -3} \frac{x^5 + 243}{x^3 + 27}$$

(a) Using l'hôpital's rule

(b) Without l'hôpital's rule

3. Find the slope of the tangent line given in polar coordinates by $r = \tan\theta$ when $\theta = \pi$.

4. Determine the intervals on which $f(x)$ is continuous, where

$$f(x) = \begin{cases} \frac{\sqrt{x+4}-2}{x} & \text{if } x < 0 \\ \\ \frac{-2}{x-8} & \text{if } x \geq 0 \end{cases}$$

5. Use the definition of the derivative to find

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x}+\sqrt{x}\right]$$

6. Approximate $\sqrt[3]{10}$ using each of the following:

   (i) Five iterations of bisection

   (ii) Local linear approximation

   (iii) Newton's method to the maximum accuracy supported by your calculator

7. Differentiate:

(a) x^π

(b) π^x

(c) x^x

(d) π^π

(e) $\frac{\sin x}{\ln x}$

(f) $\ln x \sin x$

(g) $\sin \ln x$

(h) $\ln \sin x$

8. Prove the reciprocal rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{g(x)} = -\frac{g'(x)}{g(x)^2}$$

(a) Using the definition of the derivative

(b) Using the quotient rule

9. (a) State the differentiation rule for $f(x) = \cot x$, and prove the rule using the differentiation rules for $\sin x$ and $\cos x$ and the quotient rule.

   (b) State and prove the differentiation rule for $f(x) = \cot^{-1}(x)$.

10. Find the equations of both tangent lines to the parametric curve given below at $(x, y) = (0, 0)$:

$$\begin{cases} x &= t^3 - t \\ y &= t^5 - t^2 \end{cases}$$

Check your work by using your calculator to produce a graph of the curve, and sketch a copy of the graph along with the two tangent lines.

11. Find

$$\lim_{x\to\infty} x^{\ln\left(\frac{1}{x}+1\right)}$$

12. An hourglass is formed from a double-headed cone. Each cone is $3''$ tall and $1''$ wide at the base. Suppose the top half is initially half full of sand which drains at a constant rate into the bottom half (taking exactly one hour to do so), leaving a conical depression in the surface of the sand extending all the way to the edge. The depth of the depression is always equal to the depth of the sand at the center of the pit. Determine how the depth of the sand is changing after half an hour.

13. Suppose a cardboard box with a square base is to be constructed as follows: a single piece of material is stamped out, sealed together at one edge, and has slits cut into it to form end flaps. There are two sets of flaps on each end (one folding from front to back and another folding in from the sides), and each flap covers half of the top (or bottom) of the box. If the box is to have a capacity of 54ft^3, find the dimensions of the box that minimize the amount of cardboard used in its construction. Give the dimensions both of the box when its flaps are closed and the dimensions of the flaps themselves.