Name: KEY

Each problem is worth 15 points. Show all your work for full credit; numerical or graphical estimates are unacceptable unless specifically requested. Work at least seven problems; you may work an eighth for extra credit (if you complete more than eight, I will only grade the first eight).

1. Find the derivative:

   (a) $\frac{\mathrm{d}}{\mathrm{d}x}\left[e^x + x^e + ex^e + xe^x\right]$

   (b) $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt[5]{x^3}$

   (c) $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{x^4}$

(a) $e^x + ex^{e-1} + e^2x^{e-1} + (e^x + xe^x)$

(b) $\frac{3}{5}x^{-2/5}$

(c) $\frac{(\cos x)x^4 - (\sin x)4x^3}{x^8}$

2. Find

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3 - 2x^2 - 20x + 15}{x + 4}$$

(a) Using the definition of the derivative.

(b) Using the quotient rule.

Check that your answers agree.

(a)

$$\begin{array}{r|rrrr} -4 & 1 & -2 & -20 & 15 \\ & & -4 & 24 & -16 \\ \hline & 1 & -6 & 4 & -1 \end{array}$$

$$= \frac{d}{dx}\left(x^2 - 6x + 4 - \frac{1}{x+4}\right) =$$

$$\lim_{h \to 0} \frac{\left((x+h)^2 - 6(x+h) + 4 - \frac{1}{x+h+4}\right) - \left(x^2 - 6x + 4 - \frac{1}{x+4}\right)}{h}$$

$$= \lim_{h \to 0}\left(\frac{\cancel{x^2} + \cancel{2xh} + \cancel{h^2} - \cancel{6x} - \cancel{6h} + \cancel{4} - \cancel{x^2} + \cancel{6x} - \cancel{4}}{h} + \frac{\frac{1}{x+4} - \frac{1}{x+h+4}}{h}\right)$$

$$= \lim_{h \to 0}\left(2x + h - 6 + \frac{(x+h+4) - (x+4)}{h(x+4)(x+h+4)}\right)$$

$$= \lim_{h \to 0} 2x + h - 6 + \frac{1}{(x+4)(x+h+4)} = 2x - 6 + \frac{1}{(x+4)^2}$$

(b) $$\frac{d}{dx}\left(x^2 - 6x + 4 - \frac{1}{x+4}\right) = 2x - 6 - \frac{0(x+4) - 1(1)}{(x+4)^2}$$

$$= 2x - 6 + \frac{1}{(x+4)^2}$$

✓

3. Differentiate:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{x}\tan(x)\sqrt{x}\right]$$

$e^x \tan x \sqrt{x}$

$+ e^x \sec^2 x \sqrt{x}$

$+ e^x \tan x \dfrac{1}{2\sqrt{x}}$

4. State the product rule, and prove the product rule using the definition of the derivative.

$$\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$ assuming f, g diff.able

Pf: $$\lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h} \right)$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \lim_{h \to 0} g(x+h) + f(x) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

($\because$ product law for limits)

$$= f'(x) \lim_{h \to 0} g(x+h) + f(x) g'(x)$$

$$= f'(x)g(x) + f(x)g'(x)$$

($\because$ $g(x)$ diff.able $\Rightarrow$ $g(x)$ conts).

5. State the differentiation rule for $f(x) = \cos x$, and prove this differentiation rule using the definition of the derivative. You may assume the following:

(i) $\lim_{h\to 0} \frac{\sin h}{h} = 1$

(ii) $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$

(iii) $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$\frac{d}{dx}\cos x = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h\to 0} \left(\frac{\cos x \cos h - \cos x}{h} - \frac{\sin x \sin h}{h} \right)$$

$$= \lim_{h\to 0} \left(\cos x \left(\frac{\cos h - 1}{h} \right) - \sin x \frac{\sin h}{h} \right)$$

$$= \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h} \quad (\because \text{const. mult. law})$$

$$= \cos x (0) - \sin x (1)$$

$$= -\sin x$$

6. Use the differentiation rules for $\sin x, \cos x$, and the quotient rule to prove the differentiation rule for $\cot x$.

$$\frac{d}{dx}\cot x = \frac{d}{dx}\frac{\cos x}{\sin x} = \frac{-\sin x(\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = -\csc^2 x$$

7. Find the equation for the tangent line to the graph of $f(x) = \sec x$ at the point $(0, 1)$.

$f'(x) = \sec x \tan x$

$f(0) = 1$

$f'(0) = \sec 0 \tan 0 = 0$

$y - 1 = 0(x - 0)$

$y = 1$

8. Let $f(x) = x^4 + 4x^3$.

(a) Find the equation for the tangent line to the graph of $f(x)$ at the point $(2, 48)$

(b) There is a tangent line to the graph of $f(x)$ at a point $x \neq 2$ that intersects the graph a second time at $(2, 48)$. Find the equation of this tangent line.

(a) $f'(x) = 4x^3 + 12x^2$

$f(2) = 48$

$f'(2) = 80$

$y - 48 = 80(x-2)$

$\boxed{y = 80x - 112}$

(b) $y - 48 = m(x-2)$

$x^4 - 4x^3 - 48 = (4x^3 + 12x^2)(x-2)$

$x^4 - \cancel{4x^3} - 48 = 4x^4 - \cancel{8x^3} + \cancel{12x^3} - 24x^2$

$0 = 3x^4 - 24x^2 + 48$

$= 3(x^4 - 8x^2 + 16)$

$= 3(x^2 - 4)^2$

$= 3(x-2)^2(x+2)^2$

$x = \pm 2 \qquad x = -2$

$f'(-2) = 16$

$f(-2) = -16$

$y + 16 = 16(x+2)$

$\boxed{y = 16x + 16}$

9. Suppose the position of a particle on the y-axis at time t is given by

$$D(t) = 2t^3 - 45t^2 + 300t$$

Determine when the particle is speeding up and when it is slowing down.

$D'(t) = 6t^2 - 90t + 300$

$= 6(t^2 - 15t + 50)$

$= 6(t-5)(t-10)$ $\quad t = 5, 10$

$D''(t) = 12t - 90$

$= 12(t - 7.5)$ $\quad t = 7.5$

t	$(-\infty, 5)$	$(5, 7.5)$	$(7.5, 10)$	$(10, \infty)$
$D'(t)$	+	−	−	+
$D''(t)$	−	−	+	+
spd/slow	slow	spd	slow	spd

10. Suppose the area (in m^2) inside a mushroom ring t years after germination of a spore is given by $A(t) = \pi\left(\frac{100t}{10+t}\right)^2$. Find $A(5)$ and $A'(5)$. Give units and interpret your answer.

$$A(5) = \pi\left(\frac{100(5)}{10+5}\right)^2 = \frac{10{,}000\pi}{9} \cong 3490.7 \text{ m}^2$$

$$A(t) = \pi \frac{10{,}000t^2}{t^2+20t+100}$$

$$A'(t) = \pi\left(\frac{20{,}000t(t^2+20t+100) - 10{,}000t^2(2t+20)}{(t^2+20t+100)^2}\right)$$

$$= 10{,}000\pi\left(\frac{\cancel{2t^3}+40t^2+200t-\cancel{2t^3}-20t^2}{(t+10)^4}\right)$$

$$= 10{,}000\pi\left(\frac{20t^2+200t}{(t+10)^4}\right) = \frac{200{,}000\pi t(t+10)}{(t+10)^4}$$

$$= 200{,}000\pi t/(t+10)^3$$

$$A'(5) = \cancel{10{,}000\pi\left(\frac{500+1000}{15^4}\right)} \cong 930.8 \text{ m}^2/\text{yr}$$

After five years, the mushroom ring has an area of 3490.7 m^2 and is growing at a rate of 930.8 m^2/yr.