In class I gave an example of using the squeeze theorem to show that

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=0$$

This handout goes over this example in more detail, and attempts to provide some additional motivation with respect to the thought process you would go through in finding a solution if presented with such an example. At the end of the handout are a few extra credit exercises to help you understand the method.

Recall that in order to use the squeeze theorem, we need to find two functions $g(x)$ and $h(x)$ so that:

1. $g(x) \leq \left(\sqrt{x^2+1}-x\right) \leq h(x)$,
2. $\lim_{x\to\infty} g(x) = 0$ and $\lim_{x\to\infty} h(x) = 0$, and
3. $\lim_{x\to\infty} g(x)$ and $\lim_{x\to\infty} h(x)$ are easy to calculate.

To decide what to do next, notice that what makes $\lim_{x\to\infty}\sqrt{x^2+1}-x$ difficult to compute is the presence of the square root. So our efforts should focus on finding similar functions $g(x)$ and $h(x)$ which simplify to an expression without a radical.

Let's start with $g(x)$. Removing the 1 from inside the radical makes the expression smaller and leaves a perfect square inside the square root: $g(x)=\sqrt{x^2}-x$. Simplifying this expression gives $g(x)=|x|-x$. But since we are taking the limit as x approaches infinity, we can assume $x>0$. So $g(x)=x-x=0$. This satisfies conditions 2 and 3 above, since $\lim_{x\to\infty} 0 = 0$. So it remains to show by a sequence of inequalities that condition 1 is satisfied:

$$\begin{aligned}
x^2+1 &\geq x^2 \Rightarrow\\
\sqrt{x^2+1} &\geq \sqrt{x^2} \Rightarrow\\
\sqrt{x^2+1}-x &\geq \sqrt{x^2}-x \Rightarrow\\
\sqrt{x^2+1}-x &\geq g(x)
\end{aligned}$$

To find an appropriate function $h(x)$, we should try to use a similar method: we want to replace the inside of the radical by a perfect square that is larger than x^2+1. An initial guess might be to use x^2+2x+1 inside the radical, or $h(x)=\sqrt{x^2+2x+1}-x$. Unfortunately this satisfies conditions 1 and 3 above, but not condition 2 since $\lim_{x\to\infty} h(x)\neq 0$ (you should check this). Notice that $h(x)$ simplifies to $h(x)=1$. We would like to have $h(x)$ simplify to something that satisfies conditions 2 and 3, i.e. a function for which it is easy to show that $\lim_{x\to\infty} h(x)=0$. The canonical example of such a function would be $h(x)=\frac{1}{x}$. So we just need to write $h(x)=\frac{1}{x}$ in such a way that it is clear that $h(x)\geq\sqrt{x^2+1}-x$. It is straightforward to verify that for $h(x)=\sqrt{\left(x+\frac{1}{x}\right)^2}-x$, $h(x)$ simplifies to $h(x)=\frac{1}{x}$:

$$\begin{aligned}
h(x) &= \sqrt{\left(x+\frac{1}{x}\right)^2}-x\\
&= \left|x+\frac{1}{x}\right|-x\\
&= x+\frac{1}{x}-x \quad \left(\text{since } x>0 \Rightarrow x+\frac{1}{x}>0\right)\\
&= \frac{1}{x}
\end{aligned}$$

So it remains to check that $h(x) \geq \sqrt{x^2+1} - x$. This is easier if we first rewrite $h(x)$ by expanding the square inside the radical: $h(x) = \sqrt{x^2 + 2 + \frac{1}{x^2}}$. Then we have

$$\begin{aligned}
1 + \frac{1}{x^2} &\geq 0 \,(\text{since } x > 0) \Rightarrow \\
\left(x^2+1\right) + 1 + \frac{1}{x^2} &\geq \left(x^2+1\right) + 0 \Rightarrow \\
x^2 + 2 + \frac{1}{x^2} &\geq x^2 + 1 \Rightarrow \\
\sqrt{x^2 + 2 + \frac{1}{x^2}} &\geq \sqrt{x^2+1} \Rightarrow \\
\sqrt{x^2 + 2 + \frac{1}{x^2}} - x &\geq \sqrt{x^2+1} - x \Rightarrow \\
h(x) &\geq \sqrt{x^2+1} - x
\end{aligned}$$

So we have shown that

- $g(x) \leq \sqrt{x^2+1} - x \leq h(x)$
- $\lim_{x\to\infty} g(x) = 0$
- $\lim_{x\to\infty} h(x) = 0$

So by the squeeze theorem, $\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) = 0$.

The book solves this same problem by rationalizing the numerator, which is considerably easier. Let's look at what happens when we try to use this approach to show that

$$\lim_{x\to\infty} \left(\sqrt[3]{x^3+1} - x\right) = 0$$

An initial guess might be to multiply and divide by $\sqrt[3]{x^3+1} + 1$. Unfortunately, this doesn't eliminate the cube root:

$$\begin{aligned}
\lim_{x\to\infty} \left(\sqrt[3]{x^3+1} - x\right) &= \lim_{x\to\infty} \left(\sqrt[3]{x^3+1} - x\right) \frac{\sqrt[3]{x^3+1} + x}{\sqrt[3]{x^3+1} + x} \\
&= \lim_{x\to\infty} \frac{\left(\left(x^3+1\right)^{1/3} - x\right)\left(\left(x^3+1\right)^{1/3} + x\right)}{\left(x^3+1\right)^{1/3} + x} \\
&= \lim_{x\to\infty} \frac{\left(x^3+1\right)^{2/3} - x^2}{\left(x^3+1\right)^{1/3} - x}
\end{aligned}$$

The resulting expression in the numerator isn't any easier to work with than what we started with. To determine how to rationalize the numerator when the radical is a cube root rather than a square root, let's look at why the conjugate radical works for the square root case:

$$\begin{aligned}
\sqrt{x} - 1 &= \left(\sqrt{x} - 1\right) \frac{\sqrt{x} + 1}{\sqrt{x} + 1} \\
&= \frac{\sqrt{x}^2 - 1^2}{\sqrt{x} + 1} \\
&= \frac{x - 1}{\sqrt{x} + 1}
\end{aligned}$$

Notice that the middle line is the difference of two squares, an expression of the form $A^2 - B^2$. If we reverse this process, we see that the two factors in the numerator of the first line have the form $(A - B)(A + B)$. So we are really just using the factoring formula $A^2 - B^2 = (A - B)(A + B)$ in reverse by multiplying by the missing factor $A + B$. So to get the correct conjugate radical for a cube root, we need a factoring formula for $A^3 - B^3$. The general formula for $A^n - B^n$ is $(A - B)(A^{n-1} + A^{n-2}B + A^{n-3}B^2 + \cdots + A^2B^{n-3} + AB^{n-2} + B^{n-1})$. For $n = 2$, this gives the same formula $A^2 - B^2 = (A - B)(A + B)$ as above. For $n = 3$, it gives $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$. Applying this to the problem involving the cube root above, we have $A = \sqrt[3]{x^3 + 1}$ and $B = x$. So the conjugate radical is $A^2 + AB + B^2 = \left(\sqrt[3]{x^3 + 1}\right)^2 + \left(\sqrt[3]{x^3 + 1}\right)x + x^2$. Multiplying and dividing by this expression yields the following:

$$
\begin{aligned}
\lim_{x\to\infty}\left(\sqrt[3]{x^3+1} - x\right) &= \lim_{x\to\infty}\left(\sqrt[3]{x^3+1} - x\right)\frac{\left(\sqrt[3]{x^3+1}\right)^2 + \left(\sqrt[3]{x^3+1}\right)x + x^2}{\left(\sqrt[3]{x^3+1}\right)^2 + \left(\sqrt[3]{x^3+1}\right)x + x^2} \\
&= \lim_{x\to\infty}\frac{\left(\sqrt[3]{x^3+1}\right)^3 - x^3}{\left(\sqrt[3]{x^3+1}\right)^2 + \left(\sqrt[3]{x^3+1}\right)x + x^2} \\
&= \lim_{x\to\infty}\frac{x^3 + 1 - x^3}{\left(\sqrt[3]{x^3+1}\right)^2 + \left(\sqrt[3]{x^3+1}\right)x + x^2} \\
&= \lim_{x\to\infty}\frac{1}{\left(\sqrt[3]{x^3+1}\right)^2 + \left(\sqrt[3]{x^3+1}\right)x + x^2} \\
&= \frac{1}{\left(\sqrt[3]{\infty^3+1}\right)^2 + \left(\sqrt[3]{\infty^3+1}\right)\infty + \infty^2} \\
&= \frac{1}{\infty} \\
&= 0
\end{aligned}
$$

Note that this is considerably more work than rationalizing the numerator was for the square root problem. Now let's try the cube root problem using the squeeze theorem. Since we already know that $g(x) = 0$ and $h(x) = \frac{1}{x}$ work in the square root case, we may as well try those first in the cube root problem. Now

$$
\begin{aligned}
x^3 + 1 &\geq x^3 \Rightarrow \\
\sqrt[3]{x^3 + 1} &\geq \sqrt[3]{x^3} \Rightarrow \\
\sqrt[3]{x^3 + 1} &\geq x \Rightarrow \\
\sqrt[3]{x^3 + 1} - x &\geq x - x \Rightarrow \\
\sqrt[3]{x^3 + 1} - x &\geq 0 \Rightarrow \\
\sqrt[3]{x^3 + 1} - x &\geq g(x)
\end{aligned}
$$

As before, it is clear that $\lim_{x\to\infty} g(x) = 0$. Now instead of writing $h(x) = \frac{1}{x}$ as $\sqrt{\left(x + \frac{1}{x}\right)^2} - x$, let's use the corresponding expression involving cube roots: $h(x) = \sqrt[3]{\left(x + \frac{1}{x}\right)^3} - x$ So we only need to show that $h(x) = \sqrt[3]{\left(x + \frac{1}{x}\right)^3} - x \geq \sqrt[3]{x^3 + 1} - x$. But if we expand the cube inside the radical using the expansion formula $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$, then

$$\begin{aligned} h(x) &= \sqrt[3]{\left(x+\frac{1}{x}\right)^3} - x \\ &= \sqrt[3]{x^3 + 3x^2\frac{1}{x} + 3x\frac{1}{x^2} + \frac{1}{x^3}} - x \\ &= \sqrt[3]{x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}} - x \end{aligned}$$

Also

$$\begin{aligned} 3x + \frac{3}{x} + \frac{1}{x^3} &\geq 1 \,(\text{since } x \to \infty \Rightarrow x > 1) \Rightarrow \\ x^3 + \left(3x + \frac{3}{x} + \frac{1}{x^3}\right) &\geq x^3 + 1 \Rightarrow \\ \sqrt[3]{x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}} &\geq \sqrt[3]{x^3+1} \Rightarrow \\ \sqrt[3]{x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}} - x &\geq \sqrt[3]{x^3+1} - x \Rightarrow \\ h(x) &\geq \sqrt[3]{x^3+1} - x \end{aligned}$$

But we also have

$$\begin{aligned} h(x) &= \sqrt[3]{\left(x+\frac{1}{x}\right)^3} - x \\ &= x + \frac{1}{x} - x \\ &= \frac{1}{x} \end{aligned}$$

So clearly $\lim_{x\to\infty} h(x) = 0$. So by the squeeze theorem,

$$\lim_{x\to\infty} \left(\sqrt[3]{x^3+1} - x\right) = 0$$

Exercises (5 pts each):

1. Use rationalizing the numerator to show that

$$\lim_{x\to\infty} \sqrt[5]{x^5+1} - x = 0$$

(Hint: use the factoring formula $A^5 - B^5 = (A-B)\left(A^4 + A^3B + A^2B^2 + AB^3 + B^4\right)$.)

2. Repeat the above problem, but this time use the squeeze theorem. (Hint: use the expansion formula $(A+B)^5 = A^5 + 5A^4B + 10A^3B^2 + 10A^2B^3 + 5AB^4 + B^5$.)