Name: KEY

Each problem is worth the indicated number of points. Work four of the first six problems and one of the last two; you may solve one additional problem for extra credit (if you work more than six, I will only grade the first six). Show all your work for full credit (excluding arithmetic); numerical or graphical estimates are unacceptable unless specifically requested.

1. (15 pts) Use Newton's method to approximate $\sqrt[3]{2}$. Show your initial guess x_0, show how x_1 is calculated from x_0, and then show each consecutive x_n until the method converges to the maximum accuracy supported by your calculator.

$x_0 = 1.2$ $\left(= 5/4;\ (5/4)^3 = 5^3/4^3 = 125/64 \approx 128/64 = 2 \right)$

$f(x) = x^3 - 2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.2 - \frac{1.2^3 - 2}{3(1.2)^2} = 1.262962963$$

$x_2 = 1.259928371$

$x_3 = 1.25992105$

$x_4 = x_3$

$x_0 = 1.1$
$x_1 = 1.284297521$
$x_2 = 1.260380798$
$x_3 = 1.259921218$
$x_4 = 1.25992105$
$x_5 = x_4$

$x_0 = 3$
$x_1 = 2.074074074$
$x_2 = 1.537690539$
$x_3 = 1.30746219$
$x_4 = 1.261601801$
$x_5 = 1.259923288$
$x_6 = 1.25992105$
$x_7 = x_6$

$x_0 = 1$
$x_1 = 1.\bar{3}$
$x_2 = 1.263888889$
$x_3 = 1.259933493$
$x_4 = 1.25992105$
$x_5 = x_4$

$x_0 = 1.25$
$x_1 = 1.26$
$x_2 = 1.259921055$
$x_3 = 1.25992105$
$x_4 = x_3$

$x_0 = 2$
$x_1 = 1.5$
$x_2 = 1.296296296$
$x_3 = 1.260932225$
$x_4 = 1.259921861$
$x_5 = 1.25992105$
$x_6 = x_5$

$\lim_{x\to 0} \frac{e^{3x}-3e^{2x}+3e^{x}-1}{x^3}$ $0/0$ $= \lim_{x\to 0}\frac{3e^{3x}-6e^{2x}+3e^{x}}{3x^2}$ $0/0$ $= \lim_{x\to 0}\frac{9e^{3x}-12e^{2x}+3e^{x}}{6x}$

2. (15 pts) Find

$$\lim_{x\to 0}\frac{(e^x-1)^3}{x^3}$$

$0/0$ $= \lim_{x\to 0}\frac{27e^{3x}-24e^{2x}+3e^{x}}{6} = 1$

OR $= \left(\lim_{x\to 0}\frac{e^x-1}{x}\right)^3$ $0/0$ $= \left(\lim_{x\to 0}\frac{e^x}{1}\right)^3 = \left(\frac{e^0}{1}\right)^3 = 1$

OR $0/0$ $= \lim_{x\to 0}\frac{3(e^x-1)^2(e^x)}{3x^2}$ $0/0$ $= \lim_{x\to 0}\frac{3(2(e^x-1)e^{2x}+(e^x-1)^2e^x)}{6x}$

$0/0$ $= \lim_{x\to 0}\frac{3(2(e^xe^{2x}+2(e^x-1)e^{2x})+2(e^x-1)e^{2x}+(e^x-1)^2e^x)}{6}$

$= \frac{3(2(e^0e^0+2(e^0-1)e^0)+2(e^0-1)e^0+(e^0-1)^2e^0)}{6}$

$= 6/6 = 1$

3. (15 pts) Find

$$\lim_{x\to 0^+}\left(\frac{1}{x}+\ln x\right)$$

$= \lim_{x\to 0^+}\frac{1}{x}(1+x\ln x) = \lim_{x\to 0^+}\frac{1+x\ln x}{x}$

Now

$\lim_{x\to 0^+} x\ln x = 0\cdot -\infty$, so $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x}$ $-\infty/\infty$ $= \lim_{x\to 0^+}\frac{1/x}{-1/x^2}$

$= \lim_{x\to 0^+} -x = 0.$

Hence

$\lim_{x\to 0^+}\frac{1+x\ln x}{x} = \frac{1+0}{0^+} = \infty$

4. (15 pts) Find

$$\lim_{x\to 0^+} (1-x^5)^{(1/x^5)}$$

$$\lim_{x\to 0^+} (1-x^5)^{(1/x^5)} = 1^\infty$$

$$y = (1-x^5)^{(1/x^5)}$$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{1}{x^5}\ln(1-x^5) = \lim_{x\to 0^+} \frac{\ln(1-x^5)}{x^5}$$

$0/0$

$$= \lim_{x\to 0^+} \frac{\frac{-5x^4}{1-x^5}}{5x^4} = \lim_{x\to 0^+} \frac{-1}{1-x^5} = -1$$

So $\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim_{x\to 0^+} \ln y} = e^{-1} = 1/e$

5. (15 pts) Graph $f(x) = \ln(2x^2+8)$ by hand. Show the computation of all intervals of increase/decrease, concavity, local extrema, and inflection points.

$0 = f'(x) = \dfrac{4x}{2x^2+8}$

$x = 0$ CN

$f'(-1) = -$ ↘ $(-\infty, 0)$

$f'(1) = +$ ↗ $(0, \infty)$

$0 = f''(x) = \dfrac{4(2x^2+8) - 4x(4x)}{(2x^2+8)^2}$

$= \dfrac{8x^2 + 32 - 16x^2}{(2x^2+8)^2}$

$= \dfrac{-8x^2+32}{(2x^2+8)^2}$

$= \dfrac{-8(x^2-4)}{(2x^2+8)^2}$

$x = \pm 2$ poss. ips

$f''(-3) = -$ CON $(-\infty, -2)$

$f''(0) = +$ CUP $(-2, 2)$

$f''(3) = -$ CON $(2, \infty)$

$f(0) = \ln 8 \cong 2.079$ lmin, yicept

$f(-2) = \ln 16 \cong 2.773$ ips

$f(2) = \ln 16 \cong 2.773$

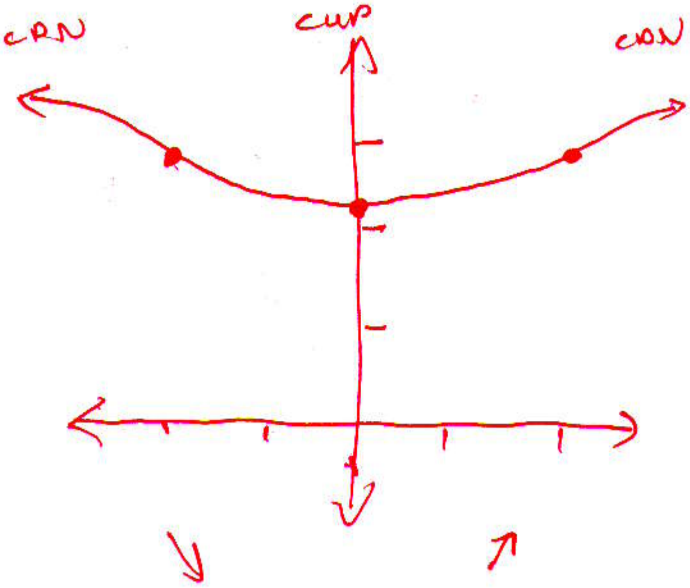
xicepts

$0 = \ln(4x^2+8)$

$e^0 = e^{\ln(4x^2+8)}$

$1 = 4x^2+8$

$-7 = 4x^2$ No sol.



6. (15 pts) Show that the area of a rectangle with fixed perimeter P is maximized when the rectangle is a square.

$A = xy$

$2x + 2y = P$

$y = \frac{P-2x}{2}$

$A = x\left(\frac{P-2x}{2}\right)$

$= \frac{Px}{2} - x^2$

$\text{dom } A = (0, P/2)$

$0 = A' = \frac{P}{2} - 2x$

$x = P/4$

$A'' = -2$

$A''(P/4) = -2 < 0$

So $x = P/4$ is lmax, only CN, and hence gmax

Then $y = \frac{P - 2(P/4)}{2}$

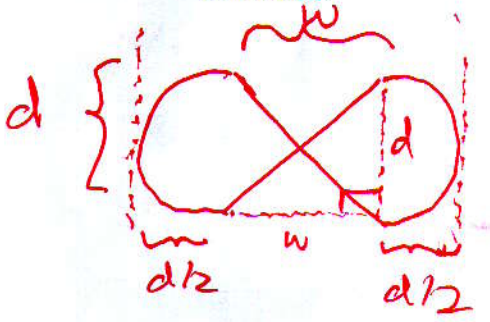
$= \frac{P - P/2}{2}$

$= \frac{P/2}{2}$

$= P/4 = x$

7. (40 pts) Suppose Bobbi and Drew want to upgrade their model train set with artificial terrain. They have decided the track should be in the shape similar to a figure-8 with two semicircles at each end connected by two straight segments that cross over each other with a bridge.

If they can only afford 250 square feet of terrain but already have all the necessary track segments (which may be assumed to have negligible width), determine the dimensions of the terrain that will maximize the length of the straight segments of the track.



$250 = d(w + 2d/2) = d(w+d)$

$250 = dw + d^2$

$w = \dfrac{250 - d^2}{d}$

$w = \dfrac{250}{d} - d$

$L = 2\sqrt{w^2 + d^2}$

maximize

$f = L^2 = 4(w^2 + d^2)$

$= 4\left(\left(\frac{250}{d} - d\right)^2 + d^2\right)$

$= 4\left(\frac{62{,}500}{d^2} - 500 + 2d^2\right)$

$f(d) = 250{,}000/d^2 - 2000 + 8d^2$

$\text{dom } f = (0, \sqrt{250})$

$(w = 0 \Rightarrow d = \sqrt{250})$

$0 = f' = -500{,}000/d^3 + 16d$

$\dfrac{500{,}000}{16} = d^4$

$d = \sqrt[4]{31{,}250}$

$= 5\sqrt[4]{50} = 5\sqrt{5}\sqrt[4]{2}$

$f'' = +1{,}500{,}000/d^4 + 16$

$f''(\sqrt[4]{31{,}250})$

$= 1{,}500{,}000/31{,}250 + 16$

$= 64 > 0$

So $d = \sqrt[4]{31{,}250}$ is min

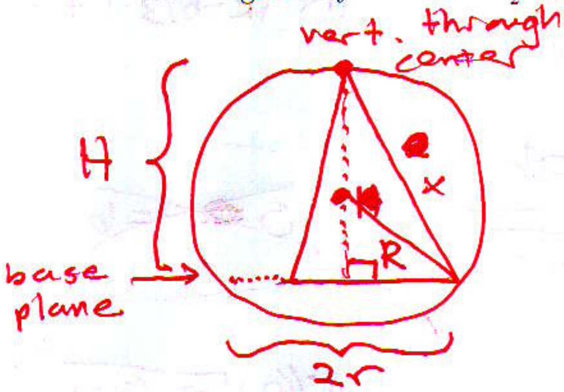
$\lim_{d \to 0} f(d) = \infty$

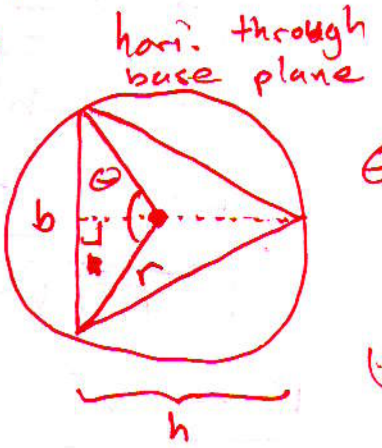
$\lim_{d \to \sqrt{250}} f(d) = 503{,}000$

$f(\sqrt[4]{31{,}250}) = 828.4$

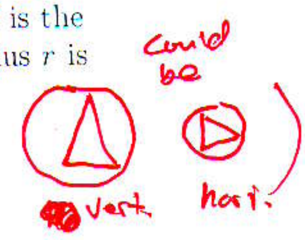
No maximum length

8. (40 pts) Consider a triangular pyramid with equilateral base contained inside a sphere of radius R. Show that the largest such pyramid is a regular tetrahedron (i.e. a triangular pyramid for which all edge lengths are equal). Use the facts that the volume of any pyramid is $V = \frac{1}{3}AH$ (where A is the area of the base and H is the height), and that the length of a chord through an angle θ in a circle of radius r is given by $C = 2r\sin\frac{\theta}{2}$.





nec. not to scale



$\theta = 2\pi/3$ since base is equilateral

$A = \frac{1}{2}bh$

$b = 2r\sin\left(\frac{2\pi/3}{2}\right) = 2r\sin(\pi/3)$

$b = 2r\frac{\sqrt{3}}{2} = r\sqrt{3}$

$r = \frac{b\sqrt{3}}{3}$

$h = r + \sqrt{r^2 - (b/2)^2}$

$= \frac{b\sqrt{3}}{3} + \sqrt{b^2/3 - b^2/4}$

$= \frac{b\sqrt{3}}{3} + \sqrt{\frac{b^2}{12}} = \frac{b\sqrt{3}}{3} + \frac{b\sqrt{3}}{6}$

$= \frac{b\sqrt{3}}{2}$

$\cancel{\text{Also } h = \frac{b}{2}\sin\frac{\pi}{3} = \frac{b}{2}\frac{\sqrt{3}}{2} = \frac{b\sqrt{3}}{4}}$

$h = \frac{b}{2}\tan\frac{\pi}{3} = \frac{b}{2}\frac{\sqrt{3}/2}{1/2} = \frac{b\sqrt{3}}{2}$ ✓

So $A = \frac{1}{2}bh = \frac{b^2\sqrt{3}}{4} = \frac{\sqrt{3}}{4}b^2$

$H = R + \sqrt{R^2 - r^2}$

$= R + \sqrt{R^2 - b^2/3}$

So

$V = \frac{1}{3}AH$

$= \frac{1}{3}\frac{\sqrt{3}}{4}b^2\left(R + \sqrt{R^2 - b^2/3}\right)$

$= \frac{\sqrt{3}R}{12}b^2 + \frac{\sqrt{3}}{12}\sqrt{R^2b^4 - b^6/3}$

Largest A occurs when base plane intersects the equator, i.e.

$r = R$, or

$b = R\sqrt{3}$

So $\text{dom } A = (0, R\sqrt{3}]$

$$V = \frac{\sqrt{3}}{12} b^2 \left(R + \sqrt{R^2 - b^2/3}\right)$$

$$V = \frac{\sqrt{3}R}{12} b^2 + \frac{\sqrt{3}}{12}\sqrt{R^2 b^4 - b^6/3}$$

$$0 = V' = \frac{\sqrt{3}R}{6} b + \frac{\sqrt{3}}{24}\,\frac{4R^2b^3 - 2b^5}{\sqrt{R^2b^4 - b^6/3}}$$

$$0 = b\left(\frac{\sqrt{3}R}{6} + \frac{\sqrt{3}}{24}\,\frac{4R^2b^2 - 2b^4}{b^2\sqrt{R^2 - b^2/3}}\right)$$

$$0 = \frac{\sqrt{3}R}{6} + \frac{\sqrt{3}}{24}\,\frac{4R^2 - 2b^2}{\sqrt{R^2 - b^2/3}}$$

$$\frac{2b^2 - 4R^2}{\sqrt{R^2 - b^2/3}} = \frac{(\sqrt{3}/6)R}{\sqrt{3}/24} = \frac{(1/6)R}{1/24} = \frac{24R}{6} = 4R$$

$$2b^2 - 4R^2 = 4R\sqrt{R^2 - b^2/3}$$

$$\left(b^2 - 2R^2 = 2R\sqrt{R^2 - b^2/3}\right)^2$$

$$b^4 - 4R^2b^2 + \cancel{4R^4} = 4R^2\left(R^2 - b^2/3\right)$$

$$= \cancel{4R^4} - 4R^2b^2/3$$

$$b^4 - 4R^2b^2 + 4R^2b^2/3 = 0$$

$$b^2\left(b^2 - \frac{8R^2}{3}\right)$$

$\cancel{b \neq 0}$, $b = \pm\sqrt{\frac{8R^2}{3}} = \pm\sqrt{\frac{2}{3}}\,2R$

$$= \cancel{\pm}\,\frac{2\sqrt{6}R}{3}$$

So $b = \frac{2\sqrt{6}}{3}R$ is gmax

So $x = \sqrt{H^2 + r^2}$

$$= \sqrt{\left(R + \sqrt{R^2 - b^2/3}\right)^2 + (b/\sqrt{3})^2}$$

$\cancel{0 = b}$

$$= \sqrt{\left(R + \sqrt{R^2 - \left(\tfrac{2\sqrt{6}}{3}R\right)^2/3}\right)^2 + \left(\tfrac{2\sqrt{6}}{3}R\right)^2/3}$$

$$= \sqrt{\left(R + \sqrt{R^2 - 8R^2/9}\right)^2 + 8R^2/9}$$

$$= \sqrt{(R + R/3)^2 + 8R^2/9} = \sqrt{24R^2/9} = 2\sqrt{6}R/3 = b$$

$V(0) = 0$

$$V\left(\frac{2\sqrt{6}R}{3}\right) = \frac{\sqrt{3}}{12}\left(\frac{8}{3}R^2\right)\left(R + \sqrt{R^2 - \frac{8}{9}R^2}\right)$$

$$= \frac{\sqrt{3}}{12}\left(\frac{8}{3}R^2\right)\left(R + R/3\right) = \frac{\sqrt{3}\,8R^3}{27}$$

gmax

$$V(R\sqrt{3}) = \frac{\sqrt{3}}{12}3R^2\left(R + \sqrt{R^2 - 3R^2/3}\right)$$

$$= \frac{3\sqrt{3}R^3}{12} = \frac{\sqrt{3}R^3}{4}$$

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