Exam 3

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Name: KGY

Each problem is worth points. Show all your work for full credit (excluding arithmetic); numerical or graphical estimates are unacceptable unless specifically requested.

1. Find the following derivatives:

   (a) $\frac{\mathrm{d}}{\mathrm{d}x}e^{\sqrt{x}}$

   $\frac{e^{\sqrt{x}}}{2\sqrt{x}}$

   (b) $\frac{\mathrm{d}}{\mathrm{d}x}\ln x$

   $\frac{1}{x}$

   (c) $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sqrt{\cos x}}{\log_2 \cot x}$ $= \frac{f}{g}$ $\quad f' = \frac{-\sin x}{2\sqrt{\cos x}} \quad g' = \frac{-\csc^2 x}{\cot x \ln 2}$

$$\frac{f'g - fg'}{g^2} = \frac{\frac{-\sin x \log_2 \cot x}{2\sqrt{\cos x}} + \frac{\sqrt{\cos x}\csc^2 x}{\cot x \ln 2}}{(\log_2 \cot x)^2}$$

2. Find where the slope of the tangent line to the graph of the polar equation

$$r = \sin\theta, 0 \le \theta \le 2\pi$$

is horizontal and where it is vertical.

$x = \sin\theta\cos\theta$

$y = \sin^2\theta$

vert

$0 = \frac{dx}{d\theta} = \cos^2\theta - \sin^2\theta$

$\sin^2\theta = \cos^2\theta$

$\sin\theta = \pm\cos\theta$

$\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$

hori

$0 = \frac{dy}{d\theta} = 2\sin\theta\cos\theta$

$\sin\theta = 0 \quad \cos\theta = 0$

$\theta = 0, \pi/2, \pi, 3\pi/2, 2\pi$

3. State the differentiation rule for $\csc^{-1} x$, and prove this rule using implicit differentiation, trigonometric identities, and the fact that $\frac{d}{dx} \csc x = -\csc x \cot x$. (Note: when you need to take a square root, you may assume that the positive branch is correct without providing any additional justification.)

$\csc^2 \alpha = \cot^2 \alpha + 1$

$\cot \alpha = \sqrt{\csc^2 \alpha - 1}$

$y = \csc^{-1} x$

$\frac{d}{dx}(\csc y = x)$

$\cot(\csc^{-1} x) = \sqrt{\csc^2(\csc^{-1} x) - 1}$

$= \sqrt{x^2 - 1}$

$-\csc y \cot y \quad y' = 1$

$$y' = \frac{-1}{\csc y \cot y}$$

$$= \frac{-1}{\csc(\csc^{-1} x) \cot(\csc^{-1} x)}$$

$$= \frac{-1}{x\sqrt{x^2-1}}$$

4. Find the y' if

$$(x+y)^3 = (x-y)^3$$

$\cancel{x^3} + 3x^2y + \cancel{3xy^2} + y^3 =$

$\cancel{x^3} - 3x^2y + \cancel{3xy^2} - y^3$

$6x^2y + 2y^3 = 0$

$12xy + 6x^2y' + 6y^2y' = 0$

$$y' = \frac{-12xy}{6x^2+6y^2}$$

$$= \frac{-2xy}{x^2+y^2}$$

5. Use local linear approximation to estimate $\sqrt[4]{250}$. Tell whether this is an underestimate or an overestimate, and explain why (hint: determine the concavity of the graph using a second derivative). For this problem only, you must show all arithmetic performed by hand, and you will receive no credit for any work done using a calculator. You may leave your final answer in the form of a fraction.

$a = 256$

$f(x) = \sqrt[4]{x} = x^{1/4}$

$f'(x) = \frac{1}{4}x^{-3/4}$

$f(a) = 4$

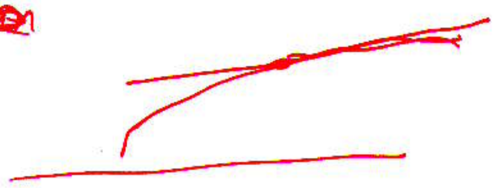
$f'(a) = \frac{1}{4}(4)^{-3} = \frac{1}{256}$

$L(x) = f'(a)(x-a) + f(a)$

$= \frac{x-256}{256} + 4$

$= \frac{x}{256} + 3$

$L(250) = \frac{250}{256} + 3$



$f''(x) = \frac{-3}{16}x^{-7/4}$

$f''(x) < 0 \ \forall x > 0 \Rightarrow f$ CDN $\Rightarrow$

$L(280)$ overest

6. Find all local and global maxima and minima of

$$f(x) = 3x^4 - 4x^3 - 36x^2$$

on $[-2, 2]$.

$f'(x) = 12x^3 - 12x^2 - 72x$

$= 12x(x^2 - x - 6)$

$= 12x(x-3)(x+2)$

$x = -2, 0, 3$

$f(-2) = -64$

$f(0) = 0$ gmax

$f(2) = -128$ gmin

7. Determine the radius of a cylinder when its volume is growing by $50\text{in}^3/\text{hr}$ and its surface area is growing by $20\text{in}^2/\text{min}$, assuming its height and diameter are always equal.

$h = 2r$

$V = \pi r^2 h$

$A = 2\pi r h + 2\pi r^2$

$V = 2\pi r^3$

$A = 4\pi r^2 + 2\pi r^2$

$= 6\pi r^2$

$V' = 50 = 6\pi (25) r'$

$r' = \frac{2}{6}\pi$

$= \frac{\pi}{3}$ in/s

$V' = 6\pi r^2 r'$

$A' = 12\pi r r'$

$$\frac{50}{20} = \frac{V'}{A'} = \frac{6\pi r^2 r'}{12\pi r r'} = \frac{r}{2}$$

$r = \frac{50}{10} = 5$ in

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