2:33: 10

Name: KEY

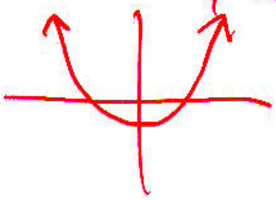
Each problem is worth 15 points. Show all your work for full credit; numerical or graphical estimates are unacceptable unless specifically requested.

1. Graph each of the following sets of parametric equations:

   (a) $\begin{cases} x = \sec t \\ y = \tan^2 t \end{cases} \quad (-\frac{\pi}{2} \le t \le \frac{\pi}{2})$

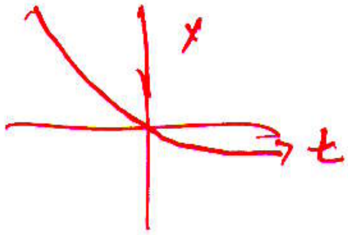
   (b) $\begin{cases} x = e^{-t} \\ y = e^{-2t} - 1 \end{cases}$

$y = x^2 - 1$



$x = e^{-t}$

$y = (e^{-t})^2 - 1 = x^2 - 1$

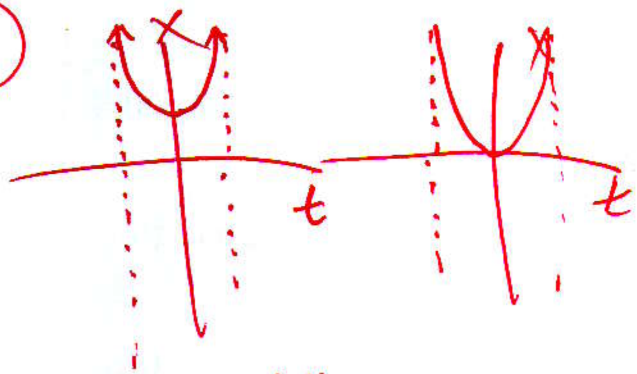


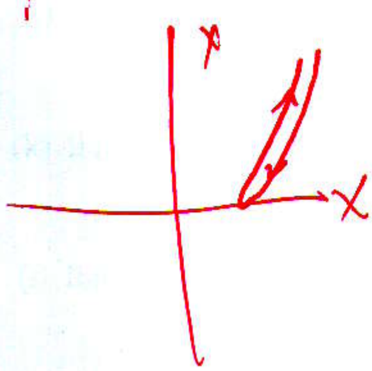
$\sec^2 t = 1 + \tan^2 t$

$x^2 = 1 + y$

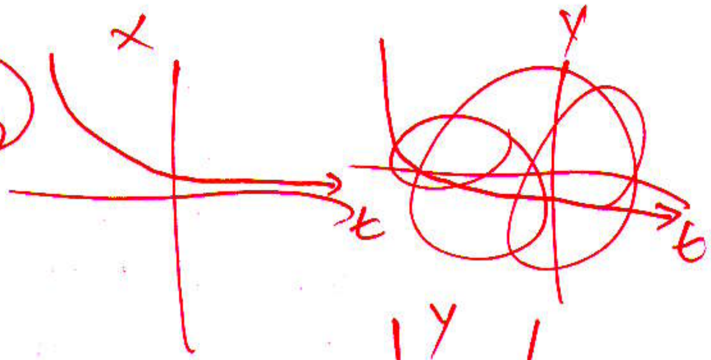
$y = x^2 - 1$

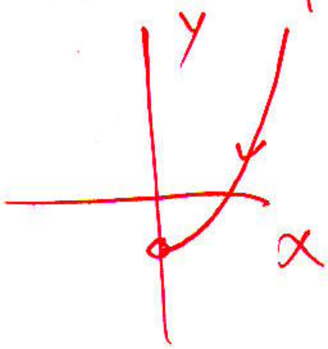
(a)





(b)





2. For the function $f(x) = \frac{x^4-16}{x^2+3x-10}$,

(a) Find $\lim_{x\to 2} f(x)$ using the limit laws.

$$\lim_{x\to 2} \frac{(x^2-4)(x^2+4)}{(x+5)(x-2)} = \lim_{x\to 2} \frac{(x+2)\cancel{(x-2)}(x^2+4)}{(x+5)\cancel{(x-2)}}$$

$$= \lim_{x\to 2} \frac{(x+2)(x^2+4)}{x+5} = \frac{4(2^2+4)}{7}$$

$$= 32/7 \cong \cancel{4.75}\ 4.57$$

(b) Check your answer from part 2a by approximating the limit with a table of values.

x	$\frac{x^4-16}{x^2+3x-10}$
1.9	~~8.649~~ 4.321
1.99	4.544
2.01	4.599
2.1	4.856

3. For the functions $f(x) = \frac{2|x|}{x} + 2$ and $g(x) = 3 - \frac{3|x|}{x}$, find the value of the limit or explain why it does not exist. For parts 3a, 3b, 3d, and 3e, you may use a graph to find the value of the limit.

(a) $\lim_{x \to 0^+} f(x)$ 4

(b) $\lim_{x \to 0^-} f(x)$

0

(c) $\lim_{x \to 0} f(x)$

DNE $a \neq b$

(d) $\lim_{x \to 0^+} g(x)$

0

(e) $\lim_{x \to 0^-} g(x)$

6

(f) $\lim_{x \to 0} g(x)$

DNE $(d) \neq e$

(g) $\lim_{x \to 0^+} (f + g)(x)$

4

(h) $\lim_{x \to 0^-} (f + g)(x)$

6

(i) $\lim_{x \to 0} (f + g)(x)$

DNE $g \neq h$

(j) $\lim_{x \to 0^+} (fg)(x)$

0

(k) $\lim_{x \to 0^-} (fg)(x)$

0

(l) $\lim_{x \to 0} (fg)(x)$

0

4. Show that

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)\sin\left(\frac{x}{\pi}\right) = 0$$

$$-1 \le \sin\frac{\pi}{x} \le 1$$

$$-\sin\frac{x}{\pi} \le \sin\frac{\pi}{x}\sin\frac{x}{\pi} \le \sin\frac{x}{\pi}$$

$$\lim_{x\to 0} \sin\frac{x}{\pi} = \sin\frac{0}{\pi} = 0 = -\sin\frac{0}{\pi} = \lim_{x\to 0} -\sin\frac{x}{\pi}$$

So by the squeeze Thm, $\lim_{x\to 0} \sin\frac{\pi}{x}\sin\frac{x}{\pi} = 0$

5. For the function $f(x)$ and each of the points a listed below, determine whether $f(x)$ is continuous at $x = a$ and explain why:

Alt:

$$f(x) = \begin{cases} \cos\left(\frac{\pi}{x}\right) & \text{if} \quad x < -1 \\ \frac{1}{x} & \text{if} \quad -1 \le x \le 1 \\ \sin\left(\frac{\pi}{x}\right) & \text{if} \quad x > 1 \end{cases}$$

(a) $a = -2$

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \cos \frac{\pi}{x} = \cos \frac{\pi}{-2} = f(-2)$$

Cont$\underline{s}$

(b) $a = -1$

Conts

$$\text{"} \lim_{x \to -1^-} \cos\left(\frac{\pi}{x}\right)$$

$$\lim_{x \to -1^-} f(x) = \cos\left(\frac{\pi}{-1}\right) = \cos(-\pi) = -1$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \frac{1}{x} = \frac{1}{-1} = -1. \quad \text{So } \lim_{x \to -1} f(x) = -1$$

$$= f(-1) = 1/-1$$

(c) $a = 0$

Not conts

$$\frac{1}{0} = f(0) \text{ undef}$$

(d) $a = 1$

Not Conts

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{1}{x} = \frac{1}{1} = 1$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sin \frac{\pi}{x} = \sin \frac{\pi}{1} = 0$$

So $\lim_{x \to 1} f(x)$ DNE

(e) $a = 2$

Conts

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \sin \frac{\pi}{x} = \sin \frac{\pi}{2} = f(2)$$

6. Find

$$\lim_{x\to\infty} \frac{4x^3 - 7x}{\sqrt{16x^6+14}}$$

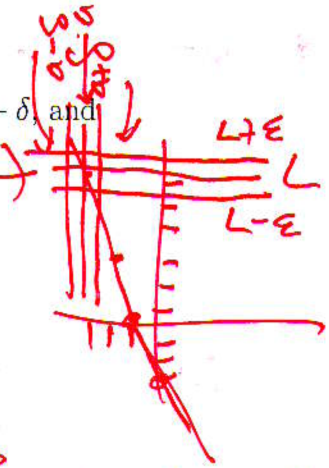
$$= \lim_{x\to\infty} \frac{4x^3 - 7x}{\sqrt{16x^6+14}} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{4 - 7/x^2}{\sqrt{\frac{16x^6+14}{x^6}}}$$

$$= \lim_{x\to\infty} \frac{4 - 7/x^7}{\sqrt{16 + 14/x^6}} = \frac{4-0}{\sqrt{16+0}} = 1$$

7. State the formal definition of the limit, and use the definition to prove that

$$\lim_{x\to -3} (-3x-3) = 6$$

Illustrate your proof with a diagram showing $f(x)$, L, a, $L+\epsilon$, $L-\epsilon$, $a+\delta$, and $a-\delta$.



$\lim_{x\to a} f(x) = L \iff \forall\ \varepsilon > 0\ \exists\ \delta > 0 \ni$
$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$

Let $\varepsilon > 0$, set $\delta = \varepsilon/3$. Then

$0 < |x-(-3)| < \delta \Rightarrow$

$-\delta < x+3 < \delta \Rightarrow$

$-\varepsilon/3 < x+3 < \varepsilon/3 \Rightarrow$

$-\varepsilon/3 - 3 < x < \varepsilon/3 - 3 \Rightarrow$

$\varepsilon + 9 > -3x > -\varepsilon + 9 \Rightarrow$

$\varepsilon + 6 > -3x - 3 > -\varepsilon + 6 \Rightarrow$

$\varepsilon > (-3x-3) - 6 > -\varepsilon$
$\Rightarrow$
$|(-3x-3) - 6| < \varepsilon$

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8. (Bonus) Find

$$\lim_{h\to 0} \frac{\sqrt[3]{27+h}-3}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt[3]{27+h}-3}{h} \cdot \frac{(27+h)^{2/3}+3(27+h)^{1/3}+9}{(27+h)^{2/3}+3(27+h)^{1/3}+9}$$

$$= \lim_{h\to 0} \frac{\cancel{27}+h-\cancel{27}}{h((27+h)^{2/3}+3(27+h)^{1/3}+9)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}((27+h)^{2/3}+3(27+h)^{1/3}+9)}$$

$$= \lim_{h\to 0} \frac{1}{(27+h)^{2/3}+3(27+h)^{1/3}+9}$$

$$= \frac{1}{9+9+9} = \frac{1}{27}$$

2:48:31