We saw in class that the correct method for rationalizing the numerator of an expression of the form

$$\frac{u - v\sqrt{w}}{D}$$

was to multiply and divide by the conjugate radical

$$u + v\sqrt{w}$$

The eliminates the radical in the numerator:

$$\left(\frac{u - v\sqrt{w}}{D}\right)\left(\frac{u + v\sqrt{w}}{u + v\sqrt{w}}\right) = \frac{u^2 - v^2 w}{D\left(v + \sqrt{w}\right)}$$

So for example, to rationalize the numerator of

$$\frac{\sqrt{x+h} - \sqrt{x}}{h}$$

we multiply and divide by the conjugate radical

$$\sqrt{x+h} + \sqrt{x}$$

yielding

$$\begin{aligned}\left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right) &= \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\ &= \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\ &= \frac{1}{\sqrt{x+h} + \sqrt{x}}\end{aligned}$$

Multiplying by the conjugate radical works because it is the missing half of the factoring formula for the difference of two squares:

$$(A - B)(A + B) = A^2 - B^2$$

So multiplying by the conjugate results in both terms in the original numerator getting squared.

We can apply this same principle to rationalize an expression involving cube roots or an arbitrary n^{th} root by finding the missing half of the factoring formula for the difference of two cubes or two n^{th} powers. For an expression like

$$u - v\sqrt[3]{w}$$

we would like to cube both terms in order to end up with

$$u^3 - \left(v\sqrt[3]{w}\right)^3 = u^3 - v^3 w$$

So we need to use the factoring formula for the difference of two cubes:

$$(A-B)\left(A^2+AB+B^2\right)=A^3-B^3$$

with $A=u$ and $B=v\sqrt[3]{w}$. So the correct conjugate radical in this case is $A^2+AB+B^2=u^2+uv\sqrt[3]{w}+\left(v\sqrt[3]{w}\right)^2$.

So for example, to rationalize the numerator of

$$\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}$$

we multiply and divide by the conjugate radical

$$\left(\sqrt[3]{x+h}\right)^2+\left(\sqrt[3]{x+h}\right)\left(\sqrt[3]{x}\right)+\left(\sqrt[3]{x}\right)^2$$

This gives

$$\frac{\left(\sqrt[3]{x+h}-\sqrt[3]{x}\right)\left(\left(\sqrt[3]{x+h}\right)^2+\left(\sqrt[3]{x+h}\right)\left(\sqrt[3]{x}\right)+\left(\sqrt[3]{x}\right)^2\right)}{h\left(\left(\sqrt[3]{x+h}\right)^2+\left(\sqrt[3]{x+h}\right)\left(\sqrt[3]{x}\right)+\left(\sqrt[3]{x}\right)^2\right)}$$

which simplifies to

$$\frac{(x+h)-x}{h\left(\left(\sqrt[3]{x+h}\right)^2+\left(\sqrt[3]{x+h}\right)\left(\sqrt[3]{x}\right)+\left(\sqrt[3]{x}\right)^2\right)}$$

Note that you may check that expanding out the numerator simplifies as stated, but this is not strictly necessary since we know that we have multiplied by the missing factor in the difference of cubes formula. Continuing, the expression above may be further simplified to

$$\begin{aligned}&\frac{h}{h\left(\left(\sqrt[3]{x+h}\right)^2+\left(\sqrt[3]{x+h}\right)\left(\sqrt[3]{x}\right)+\left(\sqrt[3]{x}\right)^2\right)}\\ =\ &\frac{1}{\left(\sqrt[3]{x+h}\right)^2+\left(\sqrt[3]{x+h}\right)\left(\sqrt[3]{x}\right)+\left(\sqrt[3]{x}\right)^2}\end{aligned}$$

Note in particular that it would be incorrect to simply adapt the method of rationalizing for square roots by changing all the squares roots to cube roots (which most people assume would work the first time they see rationalizing). Trying this out, we get

$$\begin{aligned}&\left(\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}\right)\left(\frac{\sqrt[3]{x+h}+\sqrt[3]{x}}{\sqrt[3]{x+h}+\sqrt[3]{x}}\right)\\ =\ &\frac{\left(\sqrt[3]{x+h}\right)^2-\left(\sqrt[3]{x}\right)^2}{h\left(\sqrt[3]{x+h}+\sqrt[3]{x}\right)}\end{aligned}$$

which is NOT rationalized.

Exercises:

1. (10 pts) Use the factoring formula for the difference of two fifth powers:

$$A^5 - B^5 = (A - B)\left(A^4 + A^3B + A^2B^2 + AB^3 + B^4\right)$$

to determine the correct conjugate and rationalize the numerator of

$$\frac{\sqrt[5]{x+h} - \sqrt[5]{x}}{h}$$

2. (20 pts) Find a factoring formula for the difference of two seventh powers. Verify that your factoring formula is correct by expanding the factors with distributivity, and then use your factoring formula to find the conjugate and rationalize the numerator of

$$\frac{\sqrt[7]{x+h} - \sqrt[7]{x}}{h}$$